Practice Modern Physics II, W2018, Set 2

Question 1: 2D Maxwell-Boltzman Distribution

A) In 2D the speed distribution becomes $f_{2D}(v)=\beta mv\exp\left(-\beta\frac{mv^2}{2}\right)$ with $v^2=v_x^2+v_y^2$, material. **Briefly explain** the mathematical meaning of $f_{2D}dv$, and explain why the **normalization** of f_{2D} requires $\int_0^\infty f_{2D}\,dv=1$. **Verify** by direct integration that $\int_0^\infty f_{2D}\,dv=1$.

$f_{2D}dv$ is the probability that the particle has speed in the range v to $v+dv$. The total probability must equal 1. For continuous distribution the normalization is done by integration: $\int_0^\infty f_{2D}\,dv=\int_0^\infty \beta mv\exp\left(-\beta\frac{mv^2}{2}\right)=1$.

Since

$$\int v\exp\left(-av^2\right)dv=-\frac{1}{2a}\exp\left(-av^2\right)$$

$$\int_0^\infty dvf_{2D}=\left[\exp\left(-\beta\frac{mv^2}{2}\right)\right]_0^\infty=-\exp\left(-\beta\frac{m\infty^2}{2}\right)+\exp\left(-\beta\frac{m0^2}{2}\right)=1$$

B) Find the average speed in 2D, $\overline{v}$ of a neutron at T = 300K. Data: Mass of neutron in equation sheet. HINT: $\int_0^\infty x^2\exp\left(-ax^2\right)dx=\left(\pi\right)^{1/2}/\left(4a^{3/2}\right)$.

$$\overline{v}=\int_0^\infty dvvf_{2D}=\int_0^\infty \beta mv^2\exp\left(-\beta\frac{mv^2}{2}\right)=\beta m\frac{\pi^{1/2}}{4\left(\beta m/2\right)^{1/2}}=\left(\frac{\pi k_BT}{2m}\right)^{1/2}$$

$$m=1.67\times10^{-27}kg\ ,\ \overline{v}=\left(\frac{\pi\times1.381\times10^{-23}J\bullet K^{-1}\times300K}{2\times1.67\times10^{-27}kg}\right)^{1/2}=1974\frac{m}{s}$$

C) Using $\frac{df_{2D}}{dv}=0$ find the most probable speed of a neutron at T = 300K, and compare with your answer in part B)

$$\frac{df_{2D}}{dv}=\beta m\exp\left(-\beta\frac{mv^2}{2}\right)-\beta^2m^2v^2\exp\left(-\beta\frac{mv^2}{2}\right)=0\rightarrow v^*=\left(\frac{k_BT}{m}\right)^{1/2}$$

$$v^*=\left(\frac{1.381\times10^{-23}J\bullet K^{-1}\times300K}{1.67\times10^{-27}kg}\right)^{1/2}=1575\frac{m}{s}\ ($$

Question 2 Beryllium atoms as a Degenerate Fermi Gas

In Question 3 of assignment 6, we consider helium-3, ${}^{3}_{2}He$, whose nucleus is made of two protons and one neutron, with two electrons orbiting its nucleus. We showed that its total spin can be $S_{tot} = 1/2, 3/2$, which shows that it is a fermion. We calculate the its Fermi energy: $\varepsilon_F = 6.87\times10^{-23} J \rightarrow \varepsilon_F = 4.3\times10^{-4} eV$, and Fermi temperature $T_F = 4.97K$. There are **two important points**:

Point 1: At atmospheric pressure, helium-3 is a **gas** at room temperature (~300K), but becomes a **liquid** at temperatures lower than 3.19 K. It is not a solid at very high pressure > 10 atm. This is an important point, since in order for system of **identical fermions** (electrons, helium-3, …) to behave like a **degenerate gas**, the fermions must be free to move, just like free electrons in metals. As mentioned before, the Fermi temperature of electrons in metals is usually very high > 5000K, so at room temperature they can behave like **quantum degenerate gas**. Helium-3 are fermions that remains a fluid (liquid or gas), which is free to move till T = 0, as far as we know. They do become a **quantum degenerate gas** at very low temperature.

Point 2 You may as what is the difference between a **quantum degenerate ideal gas**, and **Maxwell-Boltzman (MB) ideal** gas. For the **MB ideal gas**: the internal energy is U = Nk_BT, where N is the total number of particles, and the pressure, P, is given by PV = Nk_BT or P = nk_BT with n = N/V being the number density. For a system of **quantum degenerate ideal gas**: $U = \frac{3}{5}N\varepsilon_F$ (9.46), and $P = \frac{2}{5}n\varepsilon_F$, valid for sufficiently low temperature (which in some case is room temperature ~ 300K). As discussed in class the reason that the quantum Fermi gas has such high energy/pressure at sufficiently low temperature is that two identical fermions cannot occupy the same low energy quantum states – this is the Pauli Exclusion Principle.

For this question consider Beryllium, Z = 4, with nucleus of 4 protons and 5 neutrons, ${}^{9}_{4}Be$ (most common isotope). At atmospheric pressure, its boiling point is 2742K, and its melting point is 1560K. Hence it is a **liquid** from 1560 K to 2742K, where its mass density remains relatively constant at 1.69 g/cm^3. The atomic mass of Beryllium is $m_{Be} = 9.01u$.

A) Calculate the possible total spin of ${}^{9}_{4}Be$.

First we note that ${}^{9}_{4}Be$ has 4 electrons that will occupy the configuration $1s^2 2s^2$, and hence the electrons fully occupy two shells. Using Unsold theorem we can conclude that the total electron spin will be zero.
In term of nuclear spin, it has 4 protons (p1, p2, p3,p4) and 5 neutrons (n1,n2,n3,n4,n5). The total nuclear spin is $\vec{S}_{tot} = \vec{S}_p + \vec{S}_n$, with proton spins $\vec{S}_p = \vec{s}_{p1} + \vec{s}_{p2} + \vec{s}_{p3} + \vec{s}_{p4}$, and neutron spin $\vec{S}_n = \vec{s}_{n1} + \vec{s}_{n2} + \vec{s}_{n3} + \vec{s}_{n4} + \vec{s}_{n5}$.

We add them sequentially, starting with the proton spins, $\vec{S}_p = \vec{S}_{p1,p2} + \vec{S}_{p3,p4}$. Adding p1 and p2, $\vec{S}_{p1,p2} = \vec{s}_{p1} + \vec{s}_{p2}$ by using standard angular momentum addition rule for $s_{p1} = 1/2$, $s_{p2} = 1/2$ gives $S_{p1,p2} = 0,1$. Similarly adding p3 and p4 gives $S_{p3,p4} = 0,1$. Now we add $\vec{S}_p = \vec{S}_{p1,p2} + \vec{S}_{p3,p4}$: $S_{p1,p2} = 0, S_{p3,p4} = 0 \rightarrow S_p = 0$; $S_{p1,p2} = 0, S_{p3,p4} = 1 \rightarrow S_p = 1$; $S_{p1,p2} = 1, S_{p3,p4} = 0 \rightarrow S_p = 1$; $S_{p1,p2} = 1, S_{p3,p4} = 1 \rightarrow S_p = 0,1,2$. Hence the **possible values** are $S_p = 0,1,2$. Now we do the neutron spins $\vec{S}_n = \vec{S}_{n1,n2,n3,n4} + \vec{s}_{n5}$. Adding the first four neutron spins $\vec{S}_{n1,n2,n3,n4} = \vec{s}_{n1} + \vec{s}_{n2} + \vec{s}_{n3} + \vec{s}_{n4}$ would be identical to adding the four proton spins we just did $S_{n1,n2,n3,n4} = 0,1,2$. This would give $\vec{S}_n = \vec{S}_{n1,n2,n3,n4} + \vec{s}_{n5} \rightarrow S_n = 1/2, 3/2, 5/2$. Combining the neutron and proton spins givea $S_{tot} = 1/2, 3/2, 5/2, 7/2$, which are all half integer. Hence ${}^9_4Be$ is a fermion.

B) Assume that ${}^9_4Be$ has total spin $S_{tot} = 1/2$, and that it is a liquid. Calculate its Fermi energy and temperature. Based on your result, and the data, do you think that ${}^9_4Be$ will become a quantum degenerate gas at low temperatures. Why?

The mass density of ${}^9_4Be$ is 1.69 g/cm^3 or 1690 kg/m^3, and its mass is $m_{Be} = 9.01u \times 1.66 \times 10^{-27} kg \bullet u^{-1} = 1.49 \times 10^{-26} kg$, and its number density is $n = \frac{1690 kg \bullet m^{-3}}{1.49 \times 10^{-26} kg} = 1.13 \times 10^{29} m^{-3}$, which give the Fermi energy:

$$\varepsilon_F = \frac{h^2}{8m_{Be}} \left(\frac{3}{\pi} n \right)^{2/3} = \frac{\left(6.626 \times 10^{-34} J \bullet s\right)^2}{8\left(1.49 \times 10^{-26} kg\right)} \left(\frac{3}{\pi} 1.13 \times 10^{29} m^{-3} \right)^{2/3} = 8.35 \times 10^{-23} J$$

, which give the Fermi Temperature, $T_F = \frac{\varepsilon_F}{k_B} = \frac{8.35 \times 10^{-23} J}{1.381 \times 10^{-23} J \bullet K^{-1}} = 6.05K$, which is much less than the liquid range of ${}^9_4Be$, from 1560 K to 2742K. Hence ${}^9_4Be$ will become a solid at a much higher temperature the Fermi temperature. Consequently, it will not exhibit any properties attributed to a quantum degenerate gas (fluid).

C) At the Fermi temperature calculated in part B), estimate the **pressure** and **energy**, U, of **one mole** $N = 6.023 \times 10^{23}$ of "gas" of ${}^9_4Be$ if it is a MB gas, and it is a **quantum degenerate gas**. For this question, assume erroneously that ${}^9_4Be$ remain a gas even at such low temperature.

If its is an MB gas,

$$P = nk_B T_F = \left(1.13 \times 10^{29} m^{-3}\right)\left(1.381 \times 10^{-23} J \bullet K^{-1}\right)\left(6.05K\right) = 9.4 \times 10^6 \frac{N}{m^2} = 9.4 \times 10^6 Pa \ .$$

Noting $U = \frac{3}{2} N k_B T = \frac{3}{2}\left(6.023\times10^{23}\right)\left(1.381\times10^{-23} J \bullet K^{-1}\right)\left(6.05K\right) = 75J$.

If it were a quantum degenerate gas:

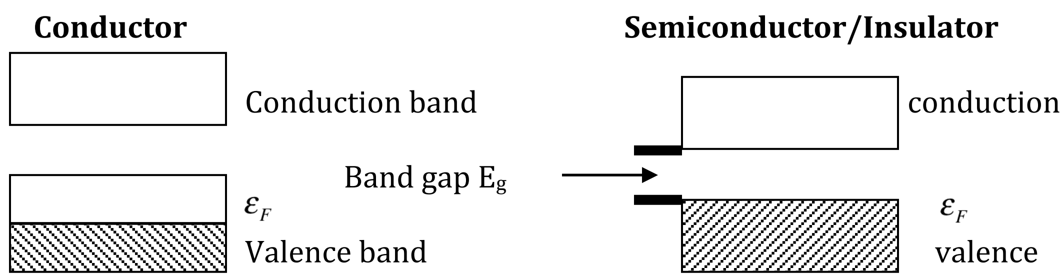
$U = \frac{3N\varepsilon_F}{5} = \frac{3}{5} 6.023\times10^{23} \times 8.35\times10^{-23} J = 30J$.

$PV = \frac{2}{3}U$, but we must calculate the volume of one mole of ${}^{9}_{4}Be$.

$n = \frac{N}{V} \rightarrow V = \frac{N}{n} = \frac{6.023\times10^{23}}{1.13\times10^{29} m^{-3}} = 5.3\times10^{-6} m^3$. This gives $P = \frac{2U}{3V} = 3.7\times10^{6} \frac{N}{m^2}$.

<u>Question 3</u> Semiconductors:

A) Using the band theory of solids explain the differences in the electrical conductivity behavior of conductors, insulators, and semiconductors. For full marks draw energy band diagram to illustrate your answer.



The above diagrams illustrate that the electron energy bands of the solid state, divided into the **allowed** valence and conduction bands. As discussed in class only electrons near the Fermi energy, ε_F can conduct into the **allowed** energy regions. For conductors (LHS) the valence band is half filled, which allow facile conduction. For semiconductor/insulator the valence band is filled, and electrons near the Fermi level ε_F must cross the band gap E_g to conduct into the conduction band. For semiconductors $E_g \sim 1eV$ and some electron (or hole) conduction occurs, thermally or due to applied voltages. For insulators $E_g > 5eV$ and virtually no electron (or hole) conduction occurs, even with an applied voltage.

B) Explain the differences between an n-type and a p-type semiconductor.

Usually pure semiconductors such as Germanium and Silicon has filled valence band gap separated from the conducting band by a gap of about 1eV. In n-type semiconductor, Silicon or Germanium is doped by material (P or As) with excess electrons, called the donor band, with energy close to the valence band, reducing the effective band gap. In p-type semiconductor, Silicon or Germanium is doped by

material (B or Al) with excess vacancy (holes), called the acceptor band, with energy close to the conduction band, reducing the effective band gap.
C) Germanium has atomic number *Z* = 32. Germanium is doped with Aluminium (Z = 13), would the resulting semiconductor be an n-type or an p-type. Briefly explain your answer.

To solve, begin by writing down the electronic configuration of silicon and aluminum, then use this to explain your answer.
Ge, Z = 32, has configuration $1s^2 2s^2 2p^6 3s^2 3p^6 3d^{10} 4s^2 4p^2$, with **four valence** electrons $4s^2 4p^2$. Aluminum has $1s^2 2s^2 2p^6 3s^2 3p^1$, with **three valence** electrons $3s^2 3p^1$, and introduces a vacancy (acceptor band) to Ge. Hence this is a p type semiconductor.

D) Repeat part C with phosphorus (Z = 15) as the dopant.
Phosphorous has $1s^2 2s^2 2p^6 3s^2 3p^3$, with **five valence** electrons $3s^2 3p^1$, and introduces an excess electron (donor band) to Ge. Hence this is a n type semiconductor.

__Question 4__ Magnetism and Superconductivity
Titanium (Z = 22), $^{48}_{22}Ti$, atomic mass 47.947 u, mass density 4.507 g/cm^3.
A) Write the electronic configuration of titanium, and determine the number of unpaired d electrons. Do you think Ti is ferromagnetic or paramagnetic. Why? **HINT:** use table 8.1 on page 275
The electronic configuration is $1s^2 2s^2 2p^6 3s^2 3p^6 3d^2 4s^2$.
There are two electrons, $3d^2$, in the 3d subshell, which should be able to accommodate 10 electrons. Hence, the two d electrons should be unpaired, and Titanium should be ferromagnetic. But in fact it is paramagnetic.

B) In fact it is Titanium is paramagnetic. Calculate the **saturation** (maximum) **magnetization**, $M_{\max}$. Hence calculate the **maximum magnetic field** that can be induced by the d spins, $B_{\max} = \mu_0 M_{\max}$, where $\mu_0 = 4\pi \times 10^{-7} T \bullet m / A$. **Hint:** see question 7 of assignment 5.
The number density of Ti is

$$n = \frac{N}{V} = \frac{\rho}{m_{Fe}} = \frac{4507 kg/m^3}{47.947u \times 1.66 \times 10^{-27} kg \bullet u^{-1}} = 5.66 \times 10^{28} \frac{particle}{m^3}.$$

As mentioned in class the magnitude magnetic moment of one unpaired electron spin is the Bohr magneton $\mu_B = \frac{e\hbar}{2m} = 9.274 \times 10^{-24} \frac{J}{T}$. Assume has 2 unpaired spins per atom, the saturation magnetization of colbalt is $M_{\max} = 2n\mu_B$. Substituting gives $M_{\max} = 2n\mu_B = 2 \times 5.66 \times 10^{28} m^{-3} \times 9.274 \times 10^{-24} J/T = 1.05 \times 10^6 A/m$.

This corresponds to a magnetic field of
$B_{max} = \mu_0 M_{max} = 4\pi \times 10^{-7} T \bullet m / A \times 1.05 \times 10^6 A / m = 1.3$T

C) Use Table 10.5 to determine the transition temperature T_c, and the critical field at zero temperature $B_c(0)$ of Titanium. For Titanium, use equation 10.44, 10.46 and 10.47 to calculate the zero temperature gap $E_g(0)$, and the energy gap at T = 0.33 K, and the critical magnetic field at T = 0.33 K, $B_c(0.33K)$.

From the table, $T_c = 0.4K$ and $B_c(0) = 5.6 \times 10^{-3} T$.

$E_g(0) = 3.54 k_B T_C = 1.95 \times 10^{-23} J = 1.22 \times 10^{-4} eV$.

$E_g(T = 0.33K) = 1.74 \times 1.22 \times 10^{-4} eV \left(1 - (0.33K / 0.4K)\right)^{1/2} = 8.8 \times 10^{-5} eV$.

$B_C(T) = B_C(0)\left(1 - (T/T_c)^2\right) = 5.6 \times 10^{-3} T\left(1 - (0.33K / 0.4K)^2\right) = 1.78 \times 10^{-3} T$.

D) A superconductor is a **perfect diamagnet** (also called the Meissner effect) such that the total magnetic field inside a superconductor is zero $B_{inside} = 0$. Give a brief physical explanation of how the Meissner effect occurs. Can the Meissner effect be explained by the spin magnetic field calculated in part B?
When an external magnetic field is applied to a superconductor, it induces a current on the surface of the superconductor that creates an internal magnetic field, in the opposite direction of the applied field, which cancels the external magnetic field. Although the magnitude of the spin-based magnetic field calculated is large enough, in the Meissner effect, the canceling field arises from a surface current, and not by aligned spin. The answer is no.

E) Using the result of part A, and the fact that Zinc, ${}^{64}_{30}Zn$ has atomic mass 63.93u to estimate the critical temperature of ${}^{64}_{30}Zn$. Compare your data with table 10.5.

Use $M^{0.5}T$ = constant.

Titanium, ${}^{48}_{22}Ti, M_{Ti} \sim 48u, T_{C,Ti} = 0.4K$

Zinc, ${}^{64}_{30}Zn, M_{Zn} \sim 64u, T_{C,Zn} = ?$

$M_{Ti}^{0.5} T_{c,Ti} = M_{Zn}^{0.5} T_{c,Zn} \rightarrow T_{c,Zn} = \frac{(48u)^{0.5}\, 0.4K}{(64u)^{0.5}} = 0.35K$, which is quite different than the 0.85 on the table. In general, the isotope effect only works for elements of the same chemical properties, such as different isotopes of mercury, as discussed on page 371 of the textbook

F) Study problem 51 chapter 10 f assignment 5