

Practice Problems for Midterm 2, 2017

Problem 1) Fermi Gas and MB gas comparison

A) Gold is a dense metal with density $1.93 \times 10^4 \text{ kg} \cdot \text{m}^{-3}$ at $T = 300\text{K}$. Calculate its number density ($n=N/V$) in m^{-3} . Assume that **a gold atom** contributes **one conducting electron**, calculate its Fermi Energy, E_F , in unit of eV . Repeat the calculation but with the assumption that **one gold atom** contributes **two conducting electrons**. Compare your answers with the experimental values of $5.53 eV$, and comment. Molar mass of gold is $197 \text{ g} \cdot \text{mol}^{-1}$

Atomic mass of gold (Au) is 197 u . We know $1\text{u} = 1.67 \times 10^{-27} \text{ kg}$, hence the mass of one gold atom is $m_{Au} = 197\text{u} \times 1.67 \times 10^{-27} \text{ kg} \cdot \text{u}^{-1} = 3.29 \times 10^{-25} \text{ kg}$. The number

density is $\frac{N}{V} = \frac{\rho}{m_{Au}} = \frac{1.93 \times 10^4 \text{ kg} \cdot \text{m}^{-3}}{3.29 \times 10^{-25} \text{ kg}} = 5.87 \times 10^{28} \text{ m}^{-3}$. Using the mass of the

electron, $m_e = 9.1 \times 10^{-31} \text{ kg}$, and assuming **one electron per gold atom**

$$\frac{N}{V} = 5.87 \times 10^{28} \text{ m}^{-3}.$$

$$E_F = \frac{h^2}{8m_e} \left(\frac{3N}{\pi V} \right)^{2/3} = \frac{(6.626 \times 10^{-34} \text{ J} \cdot \text{s})^2}{8 \times 9.1 \times 10^{-31} \text{ kg}} \left(\frac{3}{\pi} \times 5.87 \times 10^{28} \text{ m}^{-3} \right)^{2/3} = 8.83 \times 10^{-19} \text{ J}.$$

Using $1eV = 1.6 \times 10^{-19} \text{ J}$, $E_F = 8.83 \times 10^{-19} \text{ J} \div 1.6 \times 10^{-19} eV \cdot J^{-1} = 5.52 eV$. **(2 points)**

Assuming **two electron per gold atom** $\frac{N}{V} = 2 \times 5.87 \times 10^{28} \text{ m}^{-3} = 1.17 \times 10^{29} \text{ m}^{-3}$.

$$E_F = \frac{h^2}{8m_e} \left(\frac{3N}{\pi V} \right)^{2/3} = \frac{(6.626 \times 10^{-34} \text{ J} \cdot \text{s})^2}{8 \times 9.1 \times 10^{-31} \text{ kg}} \left(\frac{3}{\pi} \times 1.17 \times 10^{29} \text{ m}^{-3} \right)^{2/3} = 1.4 \times 10^{-18} \text{ J}.$$

$E_F = 1.4 \times 10^{-18} \text{ J} \div 1.6 \times 10^{-19} eV \cdot J^{-1} = 8.8 eV$. **(2 points)**

B) Use the answer of part A, to calculate the **mean Thermal energy**, $\bar{E} = U/N$, where U is the internal (mean) energy of the system, as given by the equation in the appendix.

For identical fermions, $U = 3N\varepsilon_F/5 \rightarrow \bar{E} = U/N = 3\varepsilon_F/5 = 3.32 eV$ **(2 points)**

C) Assuming that the conducting electrons can be treated a classical ideal (MB) gas, find the **mean thermal energy** at 300K . Explain the discrepancy between the values of \bar{E} obtained in part B and C.

Use the equipartition theorem the thermal energy per particle is

$$\bar{E} = \frac{U}{N} = \frac{3}{2} k_B T = \frac{3}{2} (1.381 \times 10^{-23} \text{ J} \cdot \text{K}^{-1}) (300 \text{ K}) = 6.21 \times 10^{-21} \text{ J} \text{ or } 0.038 \text{ eV. This is}$$

not surprising since classical gas are allowed to occupy the same quantum states, but **identical fermions** (such as electrons) **can not occupy the same low-energy quantum states** making the Fermi energy much greater than the thermal energy **(3 points)**

Problem 2) 2D Maxwell-Boltzman Distribution

a) In 2D the speed distribution becomes $f_{2D}(v) = \beta m v \exp\left(-\beta \frac{mv^2}{2}\right)$ with $v^2 = v_x^2 + v_y^2$, material. **Briefly explain** the mathematical meaning of $f_{2D} dv$, and explain why the **normalization** of f_{2D} requires $\int_0^\infty f_{2D} dv = 1$. **Verify** by direct integration that $\int_0^\infty f_{2D} dv = 1$.

$f_{2D} dv$ is the probability that the particle has speed in the range v to $v + dv$. The total probability must equal 1. For continuous distribution the normalization is done by integration: $\int_0^\infty f_{2D} dv = \int_0^\infty \beta m v \exp\left(-\beta \frac{mv^2}{2}\right) dv = 1$. (1 point)

Since

$$\int v \exp(-av^2) dv = -\frac{1}{2a} \exp(-av^2)$$

$$\int_0^\infty dv f_{2D} = \left[\exp\left(-\beta \frac{mv^2}{2}\right) \right]_0^\infty = -\exp\left(-\beta \frac{m\infty^2}{2}\right) + \exp\left(-\beta \frac{m0^2}{2}\right) = 1 \quad (3 \text{ points})$$

b) Find the average speed in 2D, \bar{v} of a neutron at $T = 300\text{K}$. Data: Mass of neutron in equation sheet. HINT: $\int_0^\infty x^2 \exp(-ax^2) dx = (\pi)^{1/2} / (4a^{3/2})$.

$$\bar{v} = \int_0^\infty dv v f_{2D} = \int_0^\infty \beta m v^2 \exp\left(-\beta \frac{mv^2}{2}\right) dv = \beta m \frac{\pi^{1/2}}{4(\beta m / 2)^{1/2}} = \left(\frac{\pi k_B T}{2m}\right)^{1/2}$$

$$m = 1.67 \times 10^{-27} \text{ kg}, \quad \bar{v} = \left(\frac{\pi \times 1.381 \times 10^{-23} \text{ J} \cdot \text{K}^{-1} \times 300 \text{ K}}{2 \times 1.67 \times 10^{-27} \text{ kg}}\right)^{1/2} = 1974 \frac{\text{m}}{\text{s}} \quad (3 \text{ points})$$

c) Using $\frac{df_{2D}}{dv} = 0$ find the most probable speed of a neutron at $T = 300\text{K}$, and compare with your answer in part b.

$$\frac{df_{2D}}{dv} = \beta m \exp\left(-\beta \frac{mv^2}{2}\right) - \beta^2 m^2 v^2 \exp\left(-\beta \frac{mv^2}{2}\right) = 0 \rightarrow v^* = \left(\frac{k_B T}{m}\right)^{1/2}$$

$$v^* = \left(\frac{1.381 \times 10^{-23} \text{ J} \cdot \text{K}^{-1} \times 300 \text{ K}}{1.67 \times 10^{-27} \text{ kg}}\right)^{1/2} = 1575 \frac{\text{m}}{\text{s}} \text{ (3 points)}$$

Problem 3) Vibrational Energy Level of HCl and HBr molecule.

A) Assume that the HCl molecule behaves like a harmonic oscillator with a force constant of 481 N/m. Find the energy (in eV) of its **ground** ($n = 0$) and **first excited** ($n = 1$) vibrational states. DATA: For ^1H $m_H = 1u$; ^{35}Cl $m_{Cl} = 35u$, and $1 u = 1.66 \times 10^{-27} \text{ kg}$. Note the difference between angular frequency ω (in s^{-1}) and $\nu = \omega / 2\pi$ (unit Hz).

Use $E_n = (n + 1/2)\hbar\omega$, $n = 0, 1, 2, \dots$, where $\omega = \sqrt{\frac{k}{\mu}}$, and $\mu = \frac{m_1 m_2}{m_1 + m_2}$.

$$\mu = \frac{m_H m_{Cl}}{m_H + m_{Cl}} = \frac{1u \times 35u}{1u + 35u} = 0.97u \times 1.66 \times 10^{-27} \text{ kg} \cdot u^{-1} = 1.614 \times 10^{-27} \text{ kg} \text{ (1 point)}$$

$$\omega_{HCl} = \sqrt{\frac{k}{\mu}} = \sqrt{\frac{481 \text{ N/m}}{1.614 \times 10^{-27} \text{ kg}}} = 5.46 \times 10^{14} \text{ s}^{-1} \text{ (1 point)}$$

Ground ($n = 0$)

$$E_0^{HCl} = 0.5\hbar\omega = 0.5 \times 1.055 \times 10^{-34} \text{ J} \cdot \text{s} \times 5.46 \times 10^{14} \text{ s}^{-1} = 2.88 \times 10^{-20} \text{ J} = 0.18 \text{ eV} \text{ (1 point)}$$

First Excited ($n = 1$)

$$E_1^{HCl} = 1.5\hbar\omega = 1.5 \times 1.055 \times 10^{-34} \text{ J} \cdot \text{s} \times 5.46 \times 10^{14} \text{ s}^{-1} = 8.64 \times 10^{-20} \text{ J} = 0.54 \text{ eV} \text{ (1 point)}$$

B) Bromide (^{80}Br $m_{Br} = 80u$) and Chlorine (^{35}Cl $m_{Cl} = 35u$) are **group 17** in the periodic table. In one or two sentences, use the previous sentence to justify why HBr molecule should have similar **force constant**, k , as HCl. Then assume k are the same, and calculate the **ground state vibrational energy** of HBr.

$$\text{HBr } \mu = \frac{m_H m_{Br}}{m_H + m_{Br}} = \frac{1u \times 80u}{1u + 80u} = 0.988u \times 1.66 \times 10^{-27} \text{ kg} \cdot u^{-1} = 1.64 \times 10^{-27} \text{ kg} \text{ (1 point)}$$

If we assume that HBr has the same force constant (since bonding is a chemical property that depends on electron cloud distributions, and is independent of **nuclear mass**) as HCl then

$$\omega_{HBr} = \sqrt{\frac{k}{\mu}} = \sqrt{\frac{481 \text{ N/m}}{1.64 \times 10^{-27} \text{ kg}}} = 5.42 \times 10^{14} \text{ s}^{-1} \text{ (1 point)}$$

Ground ($n = 0$)

$$E_0^{HBr} = 0.5\hbar\omega = 0.5 \times 1.055 \times 10^{-34} \text{ J} \cdot \text{s} \times 5.42 \times 10^{14} \text{ s}^{-1} = 2.86 \times 10^{-20} \text{ J} = 0.178 \text{ eV}, \text{ which is slightly smaller than the value for HCl. (1 point)}$$

C) Spectroscopic data found that the transition vibrational frequency of HCl and HBr are $\nu_{HCl} = 8.66 \times 10^{13} \text{ Hz}$ and $\nu_{HBr} = 7.68 \times 10^{13} \text{ Hz}$. Is this data consistent with the finding of part B? If the **answer** is **no**, discuss why in **no more than four sentences**.

From part A, $\nu_{HCl}^{Theory} = \omega_{HCl} / 2\pi = 5.46 \times 10^{14} \text{ s}^{-1} / 2\pi = 8.69 \times 10^{13} \text{ Hz}$

From part B, $\nu_{HBr}^{Theory} = \omega_{HBr} / 2\pi = 5.42 \times 10^{14} \text{ s}^{-1} / 2\pi = 8.63 \times 10^{13} \text{ Hz}$ **(1 point)**

Hence $\nu_{HCl}^{Theory} \approx \nu_{HBr}^{Theory}$. However, the observed values are $\nu_{HCl}^{Theory} > \nu_{HBr}^{Theory}$. This means that the assumption that the force constant of HCl and HBr are the same is **not completely valid**. **(2 points)**

Problem 4) Rotational spectroscopy The rotational energy of a diatomic molecule

is $E_{rot} = \frac{\ell(\ell+1)\hbar^2}{2I}$.

(A) In a photon absorption experiment a diatomic molecule absorbs a photon, and makes a transition from the ℓ rotational state to the $\ell + 1$ rotational state. **Show**

explicitly that the energy of the photon is $\Delta E_{photon} = \frac{(\ell+1)\hbar^2}{I}$.

For ℓ^{th} rotational state $E_{rot} = \frac{\ell(\ell+1)\hbar^2}{2I}$ and $\ell + 1^{th}$ rotational state

$$E_{rot} = \frac{(\ell+1)(\ell+2)\hbar^2}{2I} = \frac{(\ell^2 + 3\ell + 2)\hbar^2}{2I},$$

which gives $\Delta E_{photon} = \frac{(\ell^2 + 3\ell + 2)\hbar^2}{2I} - \frac{\ell(\ell+1)\hbar^2}{2I} = \frac{(2\ell+2)\hbar^2}{2I} = \frac{(\ell+1)\hbar^2}{I}$. **(2 points)**

(B) A photon of frequency $\nu = 5.1 \times 10^{11} \text{ Hz}$ is absorbed by the diatomic molecule HBr resulting in a transition from the $\ell = 0$ to the $\ell = 1$ rotational state. Use this information to determine the average bond length of the molecule. For ^1H ,

$m_H = 1 \text{ amu}$; ^{80}Br , $m_{Br} = 80 \text{ amu}$; where $1 \text{ amu} = 1.66 \times 10^{-27} \text{ kg}$.

$m_{Br} = 80 \text{ amu}$; where $1 \text{ amu} = 1.66 \times 10^{-27} \text{ kg}$.

$m_H = 1 \text{ amu} \times (1.66 \times 10^{-27} \text{ kg} / \text{amu}) = 1.66 \times 10^{-27} \text{ kg}$

$m_{Br} = 80 \text{ amu} \times (1.66 \times 10^{-27} \text{ kg} / \text{amu}) = 1.32 \times 10^{-25} \text{ kg}$ **(0.5 point)**

Reduced mass $\mu = \frac{m_H m_{Br}}{m_H + m_{Br}} = \frac{(1.66 \times 10^{-27} \text{ kg})(1.32 \times 10^{-25} \text{ kg})}{1.66 \times 10^{-27} \text{ kg} + 1.32 \times 10^{-25} \text{ kg}} = 1.64 \times 10^{-27} \text{ kg}$.

(0.5 point)

Consider the energy of photons produced by $\ell \rightarrow \ell + 1$ transition:

$$h\nu = \Delta E_{photon} = \frac{(\ell+1)\hbar^2}{I} \rightarrow \nu = \frac{(\ell+1)h}{(2\pi)^2 I}, \text{ set } \ell = 0 \text{ and use } I = \mu R^2 \text{ to obtain}$$

$$v = \frac{h}{(2\pi)^2 \mu R^2}, \text{ which gives (3 points)}$$

$$R = \sqrt{\frac{h}{(2\pi)^2 \mu v}} = \sqrt{\frac{6.626 \times 10^{-34} \text{ J} \cdot \text{s}}{(2\pi)^2 \times 1.64 \times 10^{-27} \text{ kg} \times 5.1 \times 10^{11} \text{ Hz}}} = 1.42 \times 10^{-10} \text{ m (2 points)}$$

(C) Microwave communication systems (such as wireless connection signals) operate over long distances in the atmosphere. Molecular rotational spectra are in the microwave region. Briefly explain why atmospheric gases do not absorb microwaves to any great extent.

The earth's atmosphere is filled mostly with symmetric molecules such as O_2 and N_2 , which do not possess symmetric dipole moments, and consequently cannot interact with microwave radiations. This allows microwave to propagate unhindered in the atmosphere. **(2 points)**

Problem 5

Superconductivity In the superfluid state, liquid helium-4 atoms form Bose-Einstein (BE) condensate that forms circulating vortices that do not dissipate (i.e. do not slow down). In the superconducting state, electrons in metals form BE condensate that conduct with no dissipation (the resistance of a superconductor is zero).

(A) Note that electrons are fermions, but liquid helium-4 atoms are bosons. Explain how electrons in superconducting metals form BE condensate. Your explanation (**no more than 4 sentences**) should include the energy gap E_g .

In the superconducting state, two electrons are attractive to each other due to phonon-mediated interactions. The two electrons forms a cooper pair with total spin zero, and hence are bosons that can form BE condensate. **(1.5 points)**

The bonding energy of a cooper pair is called the gap energy E_g . This is the energy needed to break the bond between a cooper pair. **(1.5 points)**

(B) The critical temperature of the superconducting state of Vanadium is $T_c = 5.4K$ and its zero temperature energy gap is $E_g(0) = 15.8 \times 10^{-4} eV$. What fraction of electrons in Vanadium is in the superconducting state at temperature $T = 3.0K$? What is the energy gap at $T = 3.0K$? **Note:** assume superconductors obey the same condensate fraction equation as superfluid helium – see equations sheet !!

Vanadium is in the superconducting state at temperature $T = 3.0K$? What is the energy gap at $T = 3.0K$? **Note:** assume superconductors obey the same condensate fraction equation as superfluid helium – see equations sheet !!

Use $F = 1 - (T / T_c)^{3/2} = 1 - \left(\frac{3.0K}{5.4K}\right)^{3/2} = 0.58$ (**This equation is not part of 2018W**

midterm 2) so 0.58 of free electrons are in superconducting state.

$$E_g(T = 3) = 1.74E_g(0)\left(1 - (T / T_c)\right)^{1/2} = 1.74 \times 15.8 \times 10^{-4} eV \left(1 - (3K / 5.4K)\right)^{1/2}$$

$$= 1.83 \times 10^{-3} eV$$

(2 points)

(C) The **isotope effect** equation states that $M^{0.5}T_c \equiv \text{constant}$, where M is the atomic mass of the element. In no more than **four sentences** explain the **underlying physical basis** of this equation.

Two negatively charge electrons are electrically repulsive, so there must be an attractive force that mediates the “bonding” of two electrons to form a Cooper pair. As discussed this force is due to **lattice vibration** that creates quantum modes called phonons. The **lattice vibration effect** is **greater** for **lighter atoms**, which has smaller atomic mass, M . Hence the critical temperature, T_c , is higher (superconducting state occurs at higher temperature) for lighter elements, and

$$T_c \propto M^{-0.5}. \text{ (2 points)}$$

Using $M^{0.5}T_c \equiv \text{constant} \rightarrow M_V^{0.5}T_{c,V} = M_{Hg}^{0.5}T_{c,Hg}$, where $V \equiv$ Vanadium, $Hg \equiv$ Mercury.

$$T_{c,Hg} = \left(M_V / M_{Hg}\right)^{0.5} T_{c,V} = (50u / 201u)^{1/2} 5.4K = 2.7K \text{ (2 points)}$$

The calculated value is significantly smaller than actual experimental value. However it is still consistent with the isotope effect that predicts a lower T_c for heavier elements

(1 point)

(D) Vanadium has atomic mass $m_V = 50u$, and Mercury (Hg) has atomic mass $m_{Hg} = 201u$. Use this to estimate the critical temperature of mercury. The actual experimental on Hg is $T_c = 4.2K$. In **no more than three sentences**, discuss whether your calculations validate the **isotope effect of superconductivity**.

Using $M^{0.5}T_c \equiv \text{constant} \rightarrow M_V^{0.5}T_{c,V} = M_{Hg}^{0.5}T_{c,Hg}$, where $V \equiv$ Vanadium, $Hg \equiv$ Mercury.

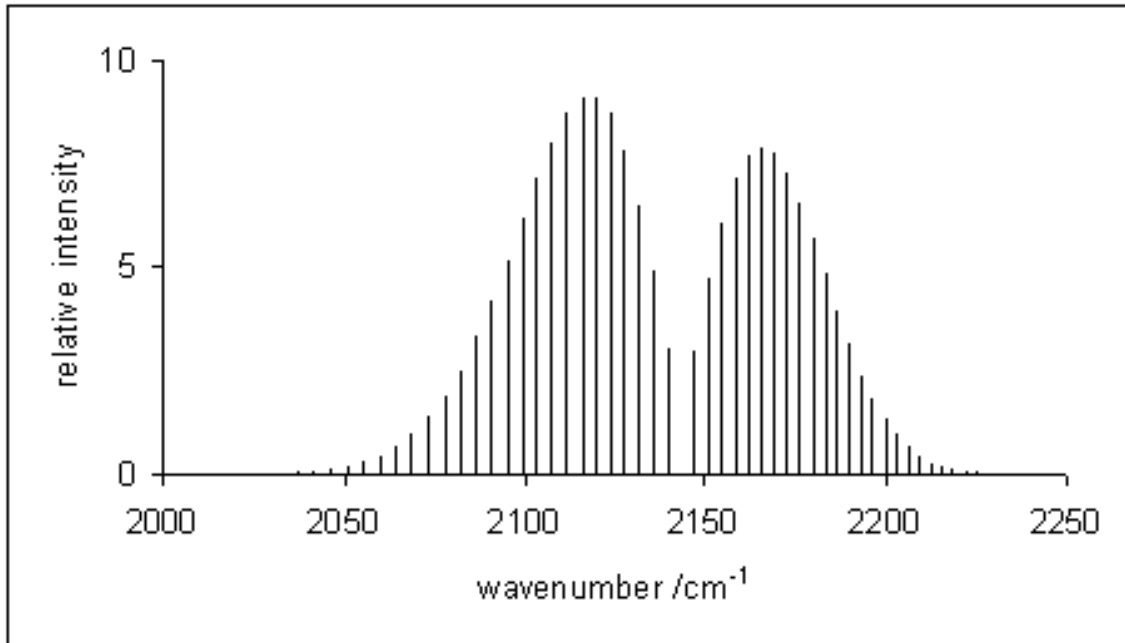
$$T_{c,Hg} = \left(M_V / M_{Hg}\right)^{0.5} T_{c,V} = (50u / 201u)^{1/2} 5.4K = 2.7K \text{ (2 points)}$$

The calculated value is significantly smaller than actual experimental value. However it is still consistent with the isotope effect that predicts a lower T_c for heavier elements

(1 point)

Problem 6

Ro-vibrational spectrum of CO is shown below:



Use the data above to calculate the bond length and bond spring constant k of CO.

Experimental data: $R = 1.13 \times 10^{-10}$ m, $k = 1860$ N/m.

Use the same method as for question 5 of assignment 5

Useful Equations

Canonical Ensemble: Constant Volume, V, particle number, N, and temperature, T. Probability of system occupying a state with energy E is proportional to the

Boltzman factor $\propto \exp(-\beta E)$,

Maxwell-Boltzman 3D speed distribution.

$F_{MB}(v)dv = 4\pi N \left(m / (2\pi k_B T) \right)^{3/2} v^2 \exp(-mv^2 / (2k_B T)) dv$; root-mean-square speed

$$v_{rms} = \sqrt{v^2} = \sqrt{\frac{3k_B T}{m}}; \text{ mean speed } \bar{v} = \sqrt{\frac{8k_B T}{\pi m}}$$

2D speed distribution $f_{2D}(v) = \beta m v \exp\left(-\beta \frac{mv^2}{2}\right)$ with $v^2 = v_x^2 + v_y^2$, $\bar{v} = \int_0^\infty dv v f_{2D}(v)$;

3D speed distribution **Ideal gas** $U = \frac{f}{2} k_B T$ f is quadratic degree of freedom;

$$PV = NK_B T = 2U / 3; U = 3Nk_B T / 2.$$

Rotational Levels: $E_\ell = \frac{L^2}{2I} = \frac{\ell(\ell+1)}{2I} \hbar^2$, $\ell = 0, 1, 2, \dots$, where $I = \mu R^2$ and $\mu = \frac{m_1 m_2}{m_1 + m_2}$.

Emission/absorption of photons: i) selection rule $\Delta\ell = \pm 1$; ii) $h\nu = \Delta E_{\ell \leftrightarrow \ell-1}$.

$\Delta E_{\ell \leftrightarrow \ell-1} = E_\ell - E_{\ell-1} = \frac{\ell \hbar^2}{I}$. Separation between adjacent lines $\frac{\hbar^2}{I}$. **Vibrational Level**

$$E_{vibr} = (n + 1/2) \hbar \omega, n = 0, 1, 2, \dots, \text{ where } \omega = \sqrt{\frac{k}{\mu}}, \text{ and } \mu = \frac{m_1 m_2}{m_1 + m_2}.$$

$\Delta E_{n+1 \leftrightarrow n} = E_{n+1} - E_n = \hbar \omega$. Absorption and Emission of photons: i) selection rule

$\Delta n = \pm 1$; ii) $h\nu = \Delta E_{n+1 \leftrightarrow n}$. **Dissociation Energy (U_0):** $E_{vibr} = (n + 1/2) \hbar \omega - U_0$.

Ro-vibrational energy is $E_{n,\ell} = \hbar \omega \left(n + \frac{1}{2} \right) + \frac{\hbar^2}{2I} \ell(\ell+1)$, with the first term

associated with vibration with quantum number, $n = 0, 1, 2, \dots$, and the second with the rotational modes with quantum numbers $\ell = 0, 1, 2, \dots$. Selection Rules

$\Delta n = n_f - n_0 = \pm 1$ and $\Delta\ell = \ell_f - \ell_0 = 0, \pm 1$, Q-branch $\Delta\ell = 0$; R-branch $\Delta\ell = +1$; P-branch

$\Delta\ell = -1$. **Wavenumber** $\bar{\nu} = 1/\lambda \rightarrow \bar{\nu} = E_{ph} / hc$, E_{ph} is **photon energy**, $E = h\nu$ and

$\lambda = c/\nu$, where $c = 2.998 \times 10^8 \text{ m/s}$.

Fermi-Dirac(FD) $f_{FD} = 1 / (e^{(\epsilon - \mu)/k_B T} + 1)$. **Bose-Einstein(BE)** $f_{BE} = 1 / (e^{(\epsilon - \mu)/k_B T} - 1)$.

Fermi Energy $\epsilon_F = (h^2 / 2m) (3N / (8\pi V))^{2/3}$; $T_F = \epsilon_F / k_B$; $u_F = \sqrt{2\epsilon_F / m}$;

$U = 3N\epsilon_F / 5$; $PV = 2U / 3$

Useful constants: Atomic mass unit (u) $1 \text{ u} = 1.66 \times 10^{-27} \text{ kg}$; electron rest mass

$m_e = 9.11 \times 10^{-31} \text{ kg}$; proton mass $m_p = 1.66 \times 10^{-27} \text{ kg}$; $e = 1.6 \times 10^{-19} \text{ C}$; $1 \text{ eV} = 1.6 \times 10^{-19} \text{ J}$;

$h = 6.62 \times 10^{-34} \text{ J}\cdot\text{s}$; $\hbar = 1.055 \times 10^{-34} \text{ J}\cdot\text{s}$; $h = 6.582 \times 10^{-16} \text{ eV}\cdot\text{s}$; Bohr

magneton $\mu_B = \frac{e\hbar}{2m} = 9.274 \times 10^{-24} \text{ J/T} ; k_B = 1.381 \times 10^{-23} \text{ J/K} = 8.617 \times 10^{-5} \text{ eV/K} .$

$$\epsilon_0 = 8.854 \times 10^{-12} \text{ C}^2 / \text{N} \cdot \text{m}^2$$

Superconductor $M^{0.5}T_c = \text{constant}$, M is the atomic weight of the substance, while T_c is the critical or transition temperature from the normal to the superconducting state. Energy Gap $E_g(0) = 3.54k_B T_c$, $E_g(T) = 1.74E_g(0)(1 - (T/T_c))^{1/2}$; critical magnetic field $B_c(T) = B_c(0)(1 - (T/T_c)^2)$.

Ferromagnetism: assume one unpaired electron can contribute magnetic moment of μ_B (up) or $-\mu_B$ (down), with $\mu_B =$. Magnetization $M = (N_+ - N_-)\mu_B$, $N_+ \equiv$ number density of up spin, $N_- \equiv$ number density of down spin. Maximum magnetization $M_{\text{max}} = N\mu_B$, N is number density of unpaired spin.

Useful Integrals: $\int_0^\infty \exp(-ax^2) dx = (\pi)^{1/2} / (2a^{1/2}) ; \int_0^\infty x^2 \exp(-ax^2) dx = (\pi)^{1/2} / (4a^{3/2}) ;$

$$\int_0^\infty x^4 \exp(-ax^2) dx = 3(\pi)^{1/2} / (8a^{5/2}) ; \int_0^\infty x \exp(-ax^2) dx = \frac{1}{2a} ; \int_0^\infty x^3 \exp(-ax^2) dx = \frac{1}{2a^2} ;$$

$$\int_0^\infty \exp(-ax) = \frac{1}{a} ; \int_0^\infty x \exp(-ax) = \frac{1}{a^2} ; \int_0^\infty x^3 \exp(-ax) = \frac{6}{a^4} .$$