**PHYS4171-Statistical Mechanics and Thermal PhysFall 2017, Assignment #6**
**Problem 1)** Problem 14 Chapter 9 **(10 points)**

From 9.29, gravitational pressure $P_{center} = \left(\frac{M}{\left(4\pi R^3/3\right)}\right) R \left(\frac{1}{2}\frac{GM}{R^2}\right)$

From 9.30, the degenerate gas pressure is $P_0 = \frac{2}{5}\left(\frac{3}{8\pi}\right)^{2/3}\frac{h^2}{2m}\left(\frac{N}{\left(4\pi R^3/3\right)}\right)^{5/3}$, where m is the mass of an electron. If we assume that the star is made up of He4 (2 protons + 2 neutrons), then the number of nucleons (protons or neutrons) in the star is about $N_n = \frac{M}{m_p}$, where M is the "effective" mass of the star, and $m_p = 1.67\times10^{-27}\,kg$ is the mass of a nucleon. Since there are 2 electrons per He4, the number of elcctrons in the star is $N = \frac{N_n}{2} = \frac{M}{2m_p}$, which gives $P_0 = \frac{2}{5}\left(\frac{3}{8\pi}\right)^{2/3}\frac{h^2}{2m}\left(\frac{1}{2m_p\left(4\pi R^3/3\right)}\right)^{5/3} M^{5/3}$

At equilibrium the two pressures must be equal:

$$P_0 = \frac{2}{5}\left(\frac{3}{8\pi}\right)^{2/3}\frac{h^2}{2m}\left(\frac{1}{2m_p\left(4\pi R^3/3\right)}\right)^{5/3} M^{5/3} = P_{center} = \left(\frac{M}{\left(4\pi R^3/3\right)}\right) R \left(\frac{1}{2}\frac{GM}{R^2}\right)$$

$$\frac{2}{5}\left(\frac{3}{8\pi}\right)^{2/3}\frac{h^2}{2m\left(2m_p\right)^{5/3}}\left(\frac{1}{\left(4\pi R^3/3\right)}\right)^{2/3} M^{-1/3} = \left(\frac{1}{2}\frac{G}{R}\right)$$

$R = \frac{1}{5mm_pG}\left(\frac{9h^3}{64m_p\pi^2}\right)^{2/3} M^{-1/3} \to \frac{R}{R_\odot} = C\left(\frac{M_\odot}{M}\right)^{1/3}$, where $M_\odot = 1.9891\times10^{30}\,kg$ is the mass of the sun, and $R_\odot = 6.957\times10^{8}\,m$ is the radius of the sun, and

$$C = \frac{1}{5mm_pG}\left(\frac{9h^3}{64m_p\pi^2}\right)^{2/3}\frac{1}{R_\odot M_\odot^{1/3}}.$$

$$C = \frac{1}{5\left(9.1\times10^{-31}kg\right)\left(1.67\times10^{-27}kg\right)\left(6.67\times10^{-11}m^3\bullet kg^{-1}\bullet s^{-2}\right)} \times\left(\frac{9\left(6.626\times10^{-34}J\bullet s\right)^3}{64\times1.67\times10^{-27}kg\times\pi^2}\right)^{2/3}\frac{1}{6.957\times10^{8}m\left(1.9891\times10^{30}kg\right)^{1/3}}$$

$$C = \left(5.068\times10^{-67}\right)^{-1}\left(1.7655\times10^{-50}\right)\left(1.143\times10^{-19}\right) = 3.98\times10^{-3}$$

For 40 ERiB, Table 9.3 gives, $C_{ERiB} = (R/R_\odot)(M/M_\odot)^{1/3} = (0.013)(0.447)^{1/3} = 1\times10^{-2}$

For Sirius B, Table 9.3 gives, $C_{SiB} = (R/R_\odot)(M/M_\odot)^{1/3} = (0.0073)(1.05)^{1/3} = 7.4\times10^{-3}$

At least it is to the same order of magnitude.

**Problem 2)** Problem 20 Chapter 9 **(10 points)**.

A) Equation 9.51, gives BE condensation temperature $k_B T_B = \frac{1}{\pi(2.612)^{2/3}}\frac{h^2}{2m}\left(\frac{N}{V}\right)^{2/3}$

For $^{23}Na \to m = 23\times1.67\times10^{-27}\,kg$, with $(N/V) = 10^{14} \div 1\times10^{-6}\,m^3 = 10^{20}\,m^{-3}$, which

gives $T_B = \frac{1}{\pi(1.381\times10^{-23}\,J/K)(2.612)^{2/3}}\frac{(6.626\times10^{-34}\,J\bullet s)^2}{2(3.841\times10^{-26}\,kg)}(10^{20}\,m^{-3})^{2/3}$.

$T_B^{23} = 1.49636\times10^{-6}\,K$.

B) Use equation 9.52, the number of atoms in the ground state is $n_1 = N\left(1-\left(\frac{T}{T_B}\right)^{3/2}\right)$. For

90% to be in the ground state $n_1 = 0.9N \to 0.9 = 1-\left(\frac{T}{T_B}\right)^{3/2} \to T = 0.1^{2/3}T_B = 3.2\times10^{-7}\,K$.

C) For $^{21}Na \to m = 21\times1.67\times10^{-27}\,kg$, with $(N/V) = 10^{14} \div 1\times10^{-6}\,m^3 = 10^{20}\,m^{-3}$, which

gives $T_B = \frac{1}{\pi(1.381\times10^{-23}\,J/K)(2.612)^{2/3}}\frac{(6.626\times10^{-34}\,J\bullet s)^2}{2(3.507\times10^{-26}\,kg)}(10^{20}\,m^{-3})^{2/3}$.

$T_B^{^{21}N} = 1.638871\times10^{-6}\,K$

For $^{23}Na \to m = 23\times1.67\times10^{-27}\,kg$, with $(N/V) = 10^{14} \div 1\times10^{-6}\,m^3 = 10^{20}\,m^{-3}$, which

gives $T_B = \frac{1}{\pi(1.381\times10^{-23}\,J/K)(2.612)^{2/3}}\frac{(6.626\times10^{-34}\,J\bullet s)^2}{2(3.841\times10^{-26}\,kg)}(10^{20}\,m^{-3})^{2/3}$.

$T_B^{23} = 1.49636\times10^{-6}\,K$

From 9.55, **below** the **transition temperature**:

$\langle E\rangle = N\varepsilon_0\left[1-\left(\frac{T}{T_B}\right)^{3/2}\right] + 0.770\left(\frac{T}{T_B}\right)^{3/2} Nk_B T, T < T_B$, where $\varepsilon_0$ is the ground-state energy, which if we **neglect** the **ground-state** part gives the heat capacity below the transition temperature $C_V = \left(\frac{\partial\langle E\rangle}{\partial T}\right)_{N,V} = 1.925Nk_B\left(\frac{T}{T_B}\right)^{3/2}$ , and $C_V$ **increases** with T(see fig 9.8).

**Above transition temperature**, 9.56 gives $C_V = \frac{3}{2}Nk_B\left[1+0.231\left(\frac{T_B}{T}\right)^{3/2}+\dots\right]$, and $C_v$ **decreases** with T (see Fig 9.8).

i) 100% $^{23}Na$ . The temperature of the gas is $1.3\times10^{-6}K$ , which is **below** the **transition temperature** of both isotopes, and of $^{23}Na$ . This means that the gas will be in the **BE condensate state**, and $C_v$ will **increase** with T in cases i.

ii) 50% $^{23}Na$ and 50% $^{21}Na$ . Here the number density will be altered since compared to part i, there are only half the amount of each identical bosons for the same volume: $(N/V) = 5\times10^{13} \div 1\times10^{-6}m^3 = 5\times10^{19}m^{-3}$ . This gives

$$T_B^{^{23}Na} = \frac{1}{\pi\left(1.381\times10^{-23}J/K\right)\left(2.612\right)^{2/3}}\frac{\left(6.626\times10^{-34}J\bullet s\right)^2}{2\left(3.841\times10^{-26}kg\right)}\left(5\times10^{19}m^{-3}\right)^{2/3} = 9.4\times10^{-7}K .$$

$$T_B^{^{21}Na} = \frac{1}{\pi\left(1.381\times10^{-23}J/K\right)\left(2.612\right)^{2/3}}\frac{\left(6.626\times10^{-34}J\bullet s\right)^2}{2\left(3.507\times10^{-26}kg\right)}\left(5\times10^{19}m^{-3}\right)^{2/3} = 1\times10^{-6}K .$$

In both cases the transition temperatures are above the actual temperature $1.3\times10^{-6}K$ , and the system is not in the **BE condensate state**, and $C_v$ will **decrease** with T in cases ii.

**Problem 3)** Problem 5 Chapter 10 **(10 points)**

A) Start wit equation 10.11, $\Delta E = T\Delta S - P\Delta V + \mu\Delta N$ , which means that the internal energy $E(S,V,N)$ is naturally a function of S, V, and N. If we want the Emthalpy to be a natural function S, P and N, we must eliminate $\Delta V$ , by defining the enthalpy as $H = E + PV \rightarrow \Delta H = \Delta E + P\Delta V + V\Delta P$ , which combines with $\Delta E = T\Delta S - P\Delta V + \mu\Delta N$ , gives $\Delta H = T\Delta S - P\Delta V + \mu\Delta N + P\Delta V + V\Delta P = T\Delta S + V\Delta P + \mu\Delta N \rightarrow H(S,P,N)$.

B) $H(S,P,N) \rightarrow \Delta H = \left(\frac{\partial H}{\partial S}\right)_{P,N}\Delta S + \left(\frac{\partial H}{\partial P}\right)_{S,N}\Delta P + \left(\frac{\partial H}{\partial N}\right)_{S,P}\Delta N$

Comparing with $\Delta H = T\Delta S + V\Delta P + \mu\Delta N$ ,

$\left(\frac{\partial H}{\partial S}\right)_{P,N} = T$ , $\left(\frac{\partial H}{\partial P}\right)_{S,N} = V$ , $\left(\frac{\partial H}{\partial N}\right)_{S,P} = \mu$ .

**Problem 4)** Problem 6 Chapter 10 **(10 points)**

Read section 6.2 on photon, where the density of state of EM modes in the frequency range $v$ to $v+dv$ is $D_{EM}dv=\left(\frac{8\pi}{c^3}\right)Vv^2dv$ (equation 6.12), and using $\int_0^\infty \frac{x^3}{\exp(x)-1}dx=\frac{\pi^4}{15}$ (see equation 6.15)

A) Note that from section 7.1, equation 7.9, the **Helmholtz Free Energy** is $F=\langle E\rangle-TS$, and equation 7.10, it is $F=-k_BT\ln Z_N$, where $Z_N$ is the N-particle **partition function**. For **identical non-interacting distinguishable particles** $Z_N=Z_1^N$, where $Z_1$ is the **one-particle partition function**. In the case where the particles are **identical non-interacting indistinguishable particles** $Z_N=Z_1^N/N!$, where 1/N! is the Gibb's over-counting factor. This means that the **N-particle Helmholtz free energy** is essentially **additive**,
$F(N)=-k_BT\ln Z_1^N=\sum_{i=1}^{N}-k_BT\ln Z_1+\text{ constant}$, where it is clear that $Z_1=Z_2=Z_3...=Z_N$.

Now the partition function is the summation of all possible Boltzman factor, $\exp(-\beta\varepsilon)$, which is $Z=\sum_\varepsilon \exp(-\beta\varepsilon)$, where it is possible for two or more states to have the same energy. For **indistinguishable** and **identical** photons of a **single mode** with frequency, *v*, and energy $\varepsilon_n=nhv$, where n is the number of photons of **that mode**, the partition function is simply $Z=\sum_{n=0}^{\infty}\exp(-\beta nhv)$. This is the **well-known geometric series**,
$\sum_{n=0}^{k}r^n=\frac{1-r^{k+1}}{1-r}$, and for $-1<r<1$, $\sum_{n=0}^{\infty}r^n=\frac{1}{1-r}$, so
$Z_v=\sum_{n=0}^{\infty}\left[\exp(-\beta hv)\right]^n=\frac{1}{1-\exp(-\beta hv)}$, where we note that $\exp(-\beta hv)<1$. We assume that the system can have any number of photons, since the chemical potential of a photon system is zero $\mu=0$ (see below). In the case where there are more than one mode of photons with frequency $v_i$ and energy $n_ihv_i$, where $n_i$ is the number of photons in the i[th] mode, the partition function should be $Z=Z_1Z_2Z_3\bullet\bullet Z_i\bullet\bullet$, where $Z_i=\frac{1}{1-\exp(-\beta hv_i)}$. Note that the multiplicative property is occurs since photons do not interact, and the absence of 1/N! is explained by the distinguishability of the different modes. Hence the Helmholtz free energy of the photon system $F_{photon}=\sum_i -k_BT\ln Z_i=\sum_v -k_BT\ln\left(\frac{1}{1-\exp(-\beta hv)}\right)$. In previous treatments of free non-interacting particles discrete summations are transformed to integrals $\sum_\varepsilon(\cdots)\to\int d\varepsilon D(\varepsilon)(\cdots)$, where $D(\varepsilon)$ is the density of state per unit energy.

Here we transform $\sum_{v}(\cdots) \to \int dv D_{EM}(v)(\cdots)$, where $D_{EM}(v)$ is the mode density of state per unit frequency, where from equation 6.12 $D_{EM}dv = \left(\frac{8\pi}{c^3}\right)Vv^2dv$.

$$F_{photon} = \sum_{v} -k_BT\ln\left(\frac{1}{1-\exp(-\beta hv)}\right) \to F_{photon} = -k_BT\int_0^\infty dvD_{EM}(v)\ln\left(\frac{1}{1-\exp(-\beta hv)}\right)$$

B) $F_{photon} = -k_BT\left(\frac{8\pi}{c^3}\right)V\int_0^\infty dvv^2\ln\left(\frac{1}{1-\exp(-\beta hv)}\right)$. Integrate by parts,

$\int d(uv) = uv = \int u d\text{v} + \int \text{v} du \to \int \text{v} du = u\text{v} - \int u d\text{v}$, with du = $v^2dv$, so that $u = \frac{v^3}{3}$, and

$\text{v} = \ln\left(\frac{1}{1-\exp(-\beta hv)}\right)$, so that $d\text{v} = -\beta h\frac{\exp(-\beta hv)}{1-\exp(-\beta hv)}dv$. This gives

$$F_{photon} = -k_BT\left(\frac{8\pi}{c^3}\right)V\left\{\left[\frac{v^3}{3}\ln\left(\frac{1}{1-\exp(-\beta hv)}\right)\right]_0^\infty + \beta h\int_0^\infty dv\frac{v^3}{3}\frac{\exp(-\beta hv)}{1-\exp(-\beta hv)}\right\}.$$

$\left[\frac{v^3}{3}\ln\left(\frac{1}{1-\exp(-\beta hv)}\right)\right]_0^\infty = \frac{0^3}{3}\ln\left(\frac{1}{1-\exp(-\beta h0)}\right) - \frac{\infty^3}{3}\ln\left(\frac{1}{1-\exp(-\beta h\infty)}\right)$. It is easy to see that the first term vanishes. For the second term we note that Taylor series expansion $\ln(1-x) \approx -x$ if x is really small, and since $\exp(-\beta h\infty)$ is definitely really small

$\ln\left(\frac{1}{1-\exp(-\beta h\infty)}\right) \sim \exp(-\beta h\infty)$, and $\frac{\infty^3}{3}\ln\left(\frac{1}{1-\exp(-\beta h\infty)}\right) \sim \frac{\infty^3}{3}\exp(-\beta h\infty) = 0$,

since $\exp(-\beta h\infty)$ approaches **zero faster** than $\infty^3$ approaches **infinity**. Hence we obtain

$F_{photon} = -k_BT\left(\frac{8\pi}{c^3}\right)V\beta h\int_0^\infty dv\frac{v^3}{3}\frac{\exp(-\beta hv)}{1-\exp(-\beta hv)}$. Make the substitution $x = \beta hv$,

$$F_{photon} = -\frac{k_BT}{3}\left(\frac{8\pi}{c^3}\right)V\left(\frac{k_BT}{h}\right)^3\frac{3!}{\Gamma(4)}\int_0^\infty dx\frac{x^{4-1}}{\exp(x)-1}dx = -2k_BT\left(\frac{8\pi}{c^3}\right)V\left(\frac{k_BT}{h}\right)^3 g_4(1).$$

From appendix, $g_4(1) = \zeta(4) = \frac{\pi^4}{90}$, $F_{photon} = -2k_BT\left(\frac{8\pi}{c^3}\right)V\left(\frac{k_BT}{h}\right)^3\frac{\pi^4}{90}$.

$$F_{photon} = -k_BT\left(\frac{8\pi}{c^3}\right)V\left(\frac{k_BT}{h}\right)^3\frac{\pi^4}{45}.$$

C) Using equations 10.15, $S = -\left(\frac{\partial F_{photon}}{\partial T}\right)_{V,N} = k_B V \left(\frac{8\pi}{c^3}\right)\left(\frac{k_B T}{h}\right)^3 \frac{4\pi^4}{45}$.

Using 10.16, $P = -\left(\frac{\partial F_{photon}}{\partial V}\right)_{T,N} = k_B T \left(\frac{8\pi}{c^3}\right)\left(\frac{k_B T}{h}\right)^3 \frac{\pi^4}{45} = \frac{1}{3}\left(\frac{8\pi^5 k_B^4}{15c^3h^3}\right)T^4$.

Hence using 10.12, $E = F + TS = -k_B T\left(\frac{8\pi}{c^3}\right)V\left(\frac{k_B T}{h}\right)^3 \frac{\pi^4}{45} + Tk_B V\left(\frac{8\pi}{c^3}\right)\left(\frac{k_B T}{h}\right)^3 \frac{4\pi^4}{45}$.

$E = k_B TV\left(\frac{8\pi}{c^3}\right)\left(\frac{k_B T}{h}\right)^3 \frac{\pi^4}{15} = \left(\frac{8\pi^5 k_B^4}{15c^3h^3}\right)VT^4$, which is the same as equation 6.15. Since

$P = \frac{1}{3}\left(\frac{8\pi^5 k_B^4}{15c^3h^3}\right)T^4 \rightarrow PV = \frac{1}{3}E$, which is the same as equation 6.18.

Also from equation 7.13, $\mu = \left(\frac{\partial F_{photon}}{\partial N}\right)_{T,V} = 0$, as stated.

**Problem 5)** Problem 8 Chapter 10 **(10 points)**
Read and understand section 10.4 on the Gibb's Free Energy: $G = E - TS + PV = F + PV$;

$S = -\left(\frac{\partial G}{\partial T}\right)_{P,N}$; $V = \left(\frac{\partial G}{\partial P}\right)_{T,N}$; $\mu = \left(\frac{\partial G}{\partial N}\right)_{T,P}$; $G = N\mu$.

**A)** For $G = -k_B TN\ln\left(aT^{5/2}/P\right)$, compute the entropy.

$S = -\left(\frac{\partial G}{\partial T}\right)_{P,N} = Nk_B \ln\left(\frac{aT^{3/2}}{P}\right) + \frac{5}{2}Nk_B = Nk_B\left(\ln\left(\frac{aT^{3/2}}{P}\right) + \frac{5}{2}\right)$.

**B)** For $G = -k_B TN\ln\left(aT^{5/2}/P\right)$, compute the heat capacity at constant pressure:

$C_P = \left(\frac{\partial H}{\partial T}\right)_{P,N}$, with the **Enthalpy** $H = E + PV = E - TS + PV - TS = G + TS$. Using the

results of A), $H = -k_B TN\ln\left(\frac{aT^{5/2}}{P}\right) + Nk_B T\left(\ln\left(\frac{aT^{3/2}}{P}\right) + \frac{5}{2}\right) = \frac{5}{2}Nk_B T$. Hence

$C_P = \left(\frac{\partial H}{\partial T}\right)_{P,N} = \frac{5}{2}Nk_B$, identical to the monatomic ideal gas result of equation (1.16) and (1.18).

**C)** Using $V = \left(\frac{\partial G}{\partial P}\right)_{T,N} = -\left(\frac{\partial}{\partial P}\left\{Nk_B T\ln\left(\frac{aT^{3/2}}{P}\right)\right\}\right)_{T,N} = \frac{Nk_B T}{P} \rightarrow PV = Nk_B T$, which is the ideal gas equation.

**D)** Use $G = E - TS + PV \rightarrow E = G + TS - PV$,

$$E = -Nk_BT\ln\left(\frac{aT^{3/2}}{P}\right) + Nk_BT\left(\ln\left(\frac{aT^{3/2}}{P}\right) + \frac{5}{2}\right) - Nk_BT \rightarrow E = \frac{3}{2}Nk_BT$$ , which is the equipartition theorem for monatomic ideal gas. Hence it is clear that

$G = -Nk_BT\ln\left(\frac{aT^{3/2}}{P}\right)$ is the Gibb's free energy of a classical monatomic ideal gas.

**Problem 6) 3D Bose-Einstein (BE) Gas, at critical temperature**: In class we showed that $\frac{N}{V} = \frac{g_{3/2}(z)}{\lambda^3}$, $\frac{PV}{k_BT} = \frac{g_{5/2}(z)}{\lambda^3}$, $\langle E\rangle = \frac{3}{2}Nk_BT\frac{g_{5/2}(z)}{g_{3/2}(z)}$, with $\lambda = \left(h^2/(2\pi mk_BT)\right)^{1/2}$, and $z = \exp(\beta\mu)$. We showed that a **phase transition** occurs at critical temperature, $T_B$. Below $T_B$, a **macroscopically large number** of BE particles occupy the ground state, $\varepsilon = 0$. The phase transition can be detected by measuring the heat capacity near $T = T_B$. Calculation in class showed that:

$$\frac{C_V}{Nk_B} = \frac{15}{4}\frac{g_{5/2}(z)}{g_{3/2}(z)} - \frac{9}{4}\frac{g_{3/2}(z)}{g_{1/2}(z)}, T > T_B,$$

$$\frac{C_V}{Nk_B} = \frac{15}{4}\frac{V}{N}\frac{\zeta(5/2)}{\lambda^3}, T \le T_B$$

**A) (5 points)** Show that for $T \le T_B$, $C_V = \frac{15}{4}N_ek_B\frac{\zeta(5/2)}{\zeta(3/2)}$, where $N_e$ is the number of BE particles in the **excited** state. Explain why this relation obeys the third law of thermodynamics.

Start with $\langle E\rangle = \frac{3}{2}Nk_BT\frac{g_{5/2}(z)}{g_{3/2}(z)}$ and $\frac{N}{V} = \frac{g_{3/2}(z)}{\lambda^3}$, which can be recombine to

$\langle E\rangle = \frac{3}{2}\frac{N}{g_{3/2}(z)}k_BTg_{5/2}(z) = \frac{3Vk_BTg_{5/2}(z)}{2\lambda^{3/2}}$. For $T \le T_B$, z = 1, $g_{3/2}(1) = \zeta(3/2)$, and the number of particles in the **excited states** is $N_e = V\frac{\zeta(3/2)}{\lambda^3}$, where $\lambda = \frac{h}{\sqrt{2\pi mk_BT}}$, the rest of the particle must then be in the **ground state** $N_0 = \langle n_1\rangle = N - N_e = N\left[1 - \left(\frac{T}{T_B}\right)^{3/2}\right]$, as given by equation 9.52. Similarly For $T \le T_B$, z = 1, $g_{3/2}(1) = \zeta(3/2)$, and

$g_{5/2}(1)=\zeta(5/2)$, so $\langle E\rangle=\dfrac{3Vk_BTg_{5/2}(z)}{2\lambda^{3/2}}$. It is easy to see that $\langle E\rangle\propto T^{5/2}$, and

$C_V=\left(\dfrac{\partial\langle E\rangle}{\partial T}\right)_{N,V}=\dfrac{15Vk_Bg_{5/2}(z)}{4\lambda^{3/2}}$. But we showed that, T < $T_B$,

$N_e=V\dfrac{\zeta(3/2)}{\lambda^3}\rightarrow\dfrac{V}{\lambda^3}=\dfrac{N_e}{\zeta(3/2)}\rightarrow C_V=\dfrac{15}{4}N_ek_B\dfrac{\zeta(5/2)}{\zeta(3/2)}$.

From earlier, the number of particles in the **excited states** is $N_e=V\dfrac{\zeta(3/2)}{\lambda^3}$, with

$\lambda=\dfrac{h}{\sqrt{2\pi mk_BT}}$, so that $N_e\propto T^{3/2}$, so that as $T\rightarrow 0, N_e=0$ and $C_v$ = 0, which is one of the third law of Thermodynamics.

**B) (5 points)** Show that the heat capacity, $C_V$, is continuous at the critical temperature, T = $T_B$ – i.e. $C_V(T=T_B+0)=C_V(T=T_B-0)$!

**HINT:** Look in the appendix at T = $T_B$, z = 1, but it is possible write $z=1=\exp(-\alpha)$,

which is equivalent to $\alpha=-\dfrac{\mu}{k_BT}\rightarrow 0$, and the appendix note, $\lim_{\alpha\rightarrow 0}g_v(\exp\alpha)\approx\dfrac{\Gamma(1-v)}{\alpha^{1-v}}$ to show that $g_{1/2}(1)\rightarrow\infty$.

Start with $\dfrac{C_V}{Nk_B}=\dfrac{15}{4}\dfrac{g_{5/2}(z)}{g_{3/2}(z)}-\dfrac{9}{4}\dfrac{g_{3/2}(z)}{g_{1/2}(z)}, T>T_B$, where at T = $T_B$, z = 1, $g_v(1)=\zeta(v)$, for $v$ > 1. For $v$ = ½ , the hint states that $g_{1/2}(1)\rightarrow\infty$, so that $\dfrac{C_V}{Nk_B}=\dfrac{15}{4}\dfrac{\zeta(5/2)}{\zeta(3/2)}$ for $T=T_B+0$,

but we know that at T = $T_B$, $N=N_e$ all particles are in the excited states, which gives,

$\dfrac{C_V}{Nk_B}=\dfrac{15}{4}N_ek_B\dfrac{\zeta(5/2)}{\zeta(3/2)}$, which is the same value as for $T=T_B+0$, so $C_V$ is continuous at T = $T_B$.

**C) Graduate Students Only(10 points)**. Show that the slope of the **heat capacity derivative** ($\dfrac{dC_V}{dT}$) is **discontinuous** at T = $T_B$.

**HINT:** Use the appendix to show that:

$$\frac{1}{Nk_B}\left(\frac{\partial C_V}{\partial T}\right)_{N,V} = \begin{cases} \frac{1}{T}\left[\frac{45}{8}\frac{g_{5/2}(z)}{g_{3/2}(z)} - \frac{9}{4}\frac{g_{3/2}(z)}{g_{1/2}(z)} - \frac{27}{8}\frac{\left(g_{3/2}(z)\right)^2 g_{-1/2}(z)}{\left(g_{1/2}(z)\right)^3}\right] & \text{for } T > T_B \\ \frac{45}{8}\frac{V}{NT\lambda^3}\zeta(5/2) & \text{for } T \le T_B \end{cases}$$ , and

$\lim_{\alpha\to 0} g_v(\exp\alpha) \approx \frac{\Gamma(1-v)}{\alpha^{1-v}}$ for $g_{1/2}(1)$ and $g_{-1/2}(1)$, with $\Gamma(1/2) = \pi^{1/2}$, $\Gamma(3/2) = \pi^{1/2}/2$.

For T ≤ T$_B$, $\frac{C_V}{Nk_B} = \frac{15}{4}\frac{V}{N}\frac{\zeta(5/2)}{\lambda^3} \rightarrow \left(\frac{\partial C_V}{\partial T}\right) = \frac{45}{8}k_B\frac{V}{T}\frac{\zeta(5/2)}{\lambda^3}$, where $\left(1/\lambda^3\right) \propto T^{3/2}$.

For T ≥ T$_B$, $\frac{C_V}{Nk_B} = \frac{15}{4}\frac{g_{5/2}(z)}{g_{3/2}(z)} - \frac{9}{4}\frac{g_{3/2}(z)}{g_{1/2}(z)}$

$$\frac{1}{Nk_B}\left(\frac{\partial C_V}{\partial T}\right)_{N,V} = \frac{15}{4}\frac{1}{g_{3/2}}\left(\frac{\partial g_{5/2}}{\partial T}\right)_{V,N} - \frac{15}{4}\frac{g_{5/2}}{g_{3/2}^2}\left(\frac{\partial g_{3/2}}{\partial T}\right)_{V,N} - \frac{9}{4}\frac{1}{g_{1/2}}\left(\frac{\partial g_{3/2}}{\partial T}\right)_{V,N} + \frac{9}{4}\frac{g_{3/2}}{g_{1/2}^2}\left(\frac{\partial g_{1/2}}{\partial T}\right)_{V,N},$$

From appendix, $\left(\frac{\partial g_{3/2}}{\partial T}\right)_{N,V} = -\frac{3}{2T}g_{3/2}(z)$, $\left(\frac{\partial g_{5/2}}{\partial T}\right)_{N,V} = -\frac{3}{2T}\frac{g_{3/2}^2}{g_{1/2}}$,

$$\left(\frac{\partial g_{1/2}}{\partial T}\right)_{N,V} = -\frac{3}{2T}\frac{g_{3/2}g_{-1/2}}{g_{1/2}} \rightarrow \frac{1}{Nk_B}\left(\frac{\partial C_V}{\partial T}\right)_{N,V} = -\frac{45}{8}\frac{1}{T}\frac{g_{3/2}}{g_{1/2}} + \frac{45}{8}\frac{1}{T}\frac{g_{5/2}}{g_{3/2}} + \frac{27}{8}\frac{1}{T}\frac{g_{3/2}}{g_{1/2}} - \frac{27}{8}\frac{1}{T}\frac{g_{3/2}^2 g_{-1/2}}{g_{1/2}^3}.$$

Combining, $\frac{1}{Nk_B}\left(\frac{\partial C_V}{\partial T}\right)_{N,V} = \frac{1}{T}\left[\frac{45}{8}\frac{g_{5/2}(z)}{g_{3/2}(z)} - \frac{9}{4}\frac{g_{3/2}(z)}{g_{1/2}(z)} - \frac{27}{8}\frac{\left(g_{3/2}(z)\right)^2 g_{-1/2}(z)}{\left(g_{1/2}(z)\right)^3}\right]$.

At T = T$_B$, z = 1, we have

$\frac{1}{Nk_B}\left(\frac{\partial C_V}{\partial T}\right)_{N,V} = \frac{1}{T_B}\left[\frac{45}{8}\frac{\zeta(5/2)}{\zeta(3/2)} - \frac{9}{4}\frac{\zeta(3/2)}{g_{1/2}(1)} - \frac{27}{8}\frac{\left(\zeta(3/2)\right)^2 g_{-1/2}(1)}{\left(g_{1/2}(1)\right)^3}\right]$, and from before

we showed, $g_{1/2}(1) = \infty$, $\frac{1}{Nk_B}\left(\frac{\partial C_V}{\partial T}\right)_{N,V} = \frac{1}{T_B}\left[\frac{45}{8}\frac{\zeta(5/2)}{\zeta(3/2)} - \frac{27}{8}\frac{\left(\zeta(3/2)\right)^2 g_{-1/2}(1)}{\left(g_{1/2}(1)\right)^3}\right]$, For $v = -½$ , ½ , use $\alpha \to 0 \to g_v(\exp\alpha) \approx \frac{\Gamma(1-v)}{\alpha^{1-v}}$, $g_{1/2}(1) \approx \frac{\pi^{1/2}}{\alpha^{1/2}}$, $g_{-1/2}(1) \approx \frac{\pi^{1/2}}{2\alpha^{3/2}}$,

$\frac{(\zeta(3/2))^2 g_{-1/2}(1)}{(g_{1/2}(1))^3} = (\zeta(3/2))^2 \frac{\pi^{1/2}}{2\alpha^{3/2}} \left(\frac{\pi^{3/2}}{\alpha^{3/2}}\right)^{-1} = \frac{(\zeta(3/2))^2}{2\pi}$, which gives

approaching transition temperature, T = $T_B$ + 0 from above

$\frac{1}{Nk_B}\left(\frac{\partial C_V}{\partial T}\right)_{\substack{N,V \\ T=T_B+0}} = \frac{1}{T_B}\left[\frac{45}{8}\frac{\zeta(5/2)}{\zeta(3/2)} - \frac{27}{16}\frac{(\zeta(3/2))^2}{\pi}\right]$. From earlier,approaching transition

temperature, T = $T_B$ – 0 from below, $\left(\frac{\partial C_V}{\partial T}\right)_{\substack{N,V \\ T=T_B-0}} = \frac{45}{8} k_B V \frac{\zeta(5/2)}{\lambda^3}$, which can be combines

with $N = V\frac{\zeta(3/2)}{\lambda^3}$ (valid for T≥$T_B$), gives $\frac{1}{Nk_B}\left(\frac{\partial C_V}{\partial T}\right)_{\substack{N,V \\ T=T_B-0}} = \frac{45}{8}\frac{1}{T_B}\frac{\zeta(5/2)}{\zeta(3/2)}$.

This gives finally, $\frac{1}{Nk_B}\left(\frac{\partial C_V}{\partial T}\right)_{\substack{N,V \\ T=T_B-0}} - \frac{1}{Nk_B}\left(\frac{\partial C_V}{\partial T}\right)_{\substack{N,V \\ T=T_B+0}} = \frac{27}{16}\frac{(\zeta(3/2))^2}{T_B \pi}$ , which verifies that the

heat capacity is discontinuous at T = $T_B$.

**Problem 7) 3D Ultra-relativistic BE gas** with dispersion relation $\varepsilon = ap$, a is a constant. In this problem **neglect** the **spin** in **all calculations**.

**A) (5 points**) Show that the **density of state** is $D(\varepsilon) = V\frac{4\pi}{h^3 a^3}\varepsilon^2$.

As always start with quantum counting $\sum(..) \to \frac{V}{h^3} 4\pi p^2 dp$, with dispersion relation

$\varepsilon = ap \to dp = d\varepsilon / a$, $\sum(..) \to \frac{V}{h^3}4\pi p^2 dp = \frac{V}{h^3 a^3} 4\pi\varepsilon^2 d\varepsilon$ , and $D(\varepsilon) = V\frac{4\pi}{h^3 a^3}\varepsilon^2$.

**B) (5 points**) Show $N = \int_0^\infty d\varepsilon \frac{1}{z^{-1}\exp(\beta\varepsilon) - 1} D(\varepsilon) \to \frac{N}{V} = b\left(\frac{1}{\beta}\right)^{m_1} g_{m_2}(z)$, where b, $m_1$ and $m_2$ are constants that you are expected to determine.

$N = \frac{4\pi V}{h^3 a^3}\int_0^\infty d\varepsilon \frac{\varepsilon^2}{z^{-1}\exp(\beta\varepsilon) - 1}$, with substitution $x = \beta\varepsilon$,

$N = \frac{4\pi V}{h^3 a^3}(k_B T)^{3/2} \frac{2!}{\Gamma(3)} \int_0^\infty d\varepsilon \frac{x^{3-1}}{z^{-1}\exp(x) - 1} = \frac{8\pi V}{h^3 a^3}(k_B T)^3 g_3(z)$, $m_1 = m_2 = 3$, $b = \frac{8\pi V k_B^3}{h^3 a^3}$.

**C) (5 points)** Derive equations for the critical temperature, $T_B$, and the number of particles in the ground state, $N_0$, that is analogous to equation 9.51 and 9.52.

For $T \le T_B$, z = 1, and $g_3(1) = \zeta(3) = 1.20206$, and $N_e = \frac{8\pi V}{h^3 a^3}(k_B T)^3 \zeta(3)$, where $N_e$ is the number of atoms in the excited states, and right at the transition temperature, $T = T_B$, the total number of atoms $N = N_e$ number of atoms in the excited states:

$N = \frac{8\pi V}{h^3 a^3}(k_B T_B)^3 \zeta(3) \rightarrow T_B^3 = \left(\frac{N}{V}\right)\frac{h^3 a^3}{8\pi k_B^3}\frac{1}{\zeta(3)}$. The **transition temperature** is

$$T_B = \left(\frac{N}{8\pi\zeta(3)V}\right)^{1/3}\frac{ha}{k_B}.$$

Below the transition temperature $T < T_B$, macroscopic occupation of the ground state gives the number of particles in the ground state as $N_0 = \langle n_1 \rangle = N - N_e$, where I note that the textbook uses $\langle n_1 \rangle$, instead of the $N_0$ that I usually employed to describe the number of particles in the ground state.

Combining $N_e = \frac{8\pi V}{h^3 a^3}(k_B T)^3 \zeta(3) = NT^3\left(\frac{N}{V}\frac{h^3 a^3}{8\pi k_B^3}\frac{1}{\zeta(3)}\right)^{-1}$, with $T_B^3 = \left(\frac{N}{V}\right)\frac{h^3 a^3}{8\pi k_B^3}\frac{1}{\zeta(3)}$,

We obtain, $N_e = N\left(\frac{T}{T_B}\right)^3$, and $N_0 = N - N_e = N\left(1 - \left(\frac{T}{T_B}\right)^3\right)$.

**<u>APPENDIX</u> Bose-Einstein(BE)** function: $g_v = \frac{1}{\Gamma(v)}\int_0^\infty \frac{x^{v-1}}{z^{-1}\exp(x) - 1}$.

Expansion form, $g_v = z + \frac{z^2}{2^v} + \frac{z^3}{3^v} + \frac{z^4}{4^v} + \ldots$, $1 < z < 0$. High T (classical), $g_v \approx z$, small z.

Low temperature, $z \rightarrow 1$, $g_v(1) = \zeta(v)$ the Riemann-Zeta function.

**3D BE** gas with **dispersion relation** $\varepsilon = \frac{p^2}{2m}$, $D(\varepsilon) = \frac{2\pi V(2m)^{3/2}}{h^3}\varepsilon^{1/2}$,

$N = \int_0^\infty \frac{D(\varepsilon)d\varepsilon}{z^{-1}\exp(\beta\varepsilon) - 1} = \frac{Vg_{3/2}(z)}{\lambda^3}$, and $\frac{PV}{k_B T} = -\int_0^\infty d\varepsilon D(\varepsilon)\ln(1 - z\exp(-\beta\varepsilon)) = \frac{Vg_{5/2}(z)}{\lambda^3}$,

$\lambda = \frac{h}{\sqrt{2\pi m k_B T}}$. $\langle E \rangle = -\left(\frac{\partial q}{\partial \beta}\right)_{z,V}$, $q = \frac{PV}{k_B T}$. Heat capacity, **constant volume**,

$C_V = \left(\frac{\partial \langle E \rangle}{\partial T}\right)_{N,V}$ , **constant pressure**, $C_P = \left(\frac{\partial H}{\partial T}\right)_{N,P}$ , Enthalpy $H = \langle E \rangle + PV$ . Using above we can show $\langle E \rangle = \frac{3}{2} k_B T \frac{V}{\lambda^3} g_{5/2}(z) = \frac{3}{2} N k_B T \frac{g_{5/2}(z)}{g_{3/2}(z)}$, also $PV = \frac{2}{3}\langle E \rangle$.

Calculating $C_V$ is complicated by the fact that $\langle E \rangle$ is a function of T and z (or V), but we do not know the explicit for of the fugacity $z = \exp(\beta\varepsilon)$. Hence

$$C_V = \left(\frac{\partial \langle E \rangle}{\partial T}\right)_{N,V} = \frac{3}{2} N k_B \left[ \frac{g_{5/2}(z)}{g_{3/2}(z)} + T \frac{1}{g_{3/2}(z)} \left(\frac{\partial g_{5/2}}{\partial T}\right)_{N,V} - T \frac{g_{5/2}(z)}{g_{3/2}^2(z)} \left(\frac{\partial g_{3/2}}{\partial T}\right)_{N,V} \right].$$

We must find $\left(\frac{\partial g_{3/2}}{\partial T}\right)_{N,V}$ and $\left(\frac{\partial g_{5/2}}{\partial T}\right)_{N,V}$ . Start with $N = \frac{V g_{3/2}(z)}{\lambda^3} \rightarrow g_{3/2}(z) = \frac{N}{V}\lambda^3$, and

$\lambda^3 \propto T^{-3/2} \rightarrow \left(\frac{\partial g_{3/2}}{\partial T}\right)_{N,V} = -\frac{3}{2T} g_{3/2}(z)$. But using, $g_v(z) = z + \frac{z^2}{2^v} + \ldots \rightarrow z \frac{dg_v}{dz} = g_{v-1}$,

which gives $\left(\frac{\partial g_{3/2}}{\partial T}\right)_{N,V} = \frac{dg_{3/2}}{dz}\left(\frac{\partial z}{\partial T}\right)_{N,V} = -\frac{3}{2T} g_{3/2}(z) \rightarrow \left(\frac{\partial z}{\partial T}\right)_{N,V} = -z \frac{3}{2T} \frac{g_{3/2}}{g_{1/2}}$.

$\left(\frac{\partial g_{5/2}}{\partial T}\right)_{N,V} = \frac{dg_{5/2}}{dz}\left(\frac{\partial z}{\partial T}\right)_{N,V} = -\frac{3}{2T}\frac{g_{3/2}^2}{g_{1/2}}$, and $\left(\frac{\partial g_{1/2}}{\partial T}\right)_{N,V} = -\frac{3}{2T}\frac{g_{3/2} g_{-1/2}}{g_{1/2}}$.

Using the above relations, $\frac{C_V}{N k_B} = \frac{15}{4}\frac{g_{5/2}(z)}{g_{3/2}(z)} - \frac{9}{4}\frac{g_{3/2}(z)}{g_{1/2}(z)}, T > T_B$.

**<u>Special Behavior of BE gas at Low Temperature</u>**

At Low temperature, $z \rightarrow 1$, $g_v(z=1) = \zeta(v)$. Some values are $\zeta(2) = \frac{\pi^2}{6}$; $\zeta(4) = \frac{\pi^4}{90}$; $\zeta(6) = \frac{\pi^6}{945}$; $\zeta\left(\frac{3}{2}\right) = 2.61328$; $\zeta\left(\frac{5}{2}\right) = 1.34349$; $\zeta\left(\frac{7}{2}\right) = 1.12673$; $\zeta(3) = 1.20206$; $\zeta(5) = 1.03693$; $\zeta(7) = 1.00835$.

Some values of *v* have special behavior as $z \rightarrow 1$, or $\alpha \rightarrow 0$ $z = \exp(\beta\mu) = \exp-\alpha$.

$g_v(\exp-\alpha) = \frac{\Gamma(1-v)}{\alpha^{1-v}} + \sum_{i=0}^{\infty} \frac{(-1)^i}{i!}\zeta(v-i)\alpha^i$ , $\lim_{\alpha\rightarrow 0} g_v(\exp\alpha) \approx \frac{\Gamma(1-v)}{\alpha^{1-v}}$.