PHYS4171-Statistical Mechanics and Thermal Physics

Fall 2017 Assignment #5 Due on November 22, 2017.

Problem 1) Problem 4 Chapter 9 **(10 points)** This problem consider a system with density of state $D(\varepsilon)=\varepsilon/(\varepsilon_0^2)$.

A) To find the **Fermi energy**, ε_F, use the method of equation 9.8 and 9.9:

$$N=\int_0^{\varepsilon_F} d\varepsilon D(\varepsilon)=\frac{1}{\varepsilon_0^2}\int_0^{\varepsilon_F} d\varepsilon\varepsilon=\frac{\varepsilon_F^2}{2\varepsilon_0^2}\rightarrow\varepsilon_F=\sqrt{2N}\varepsilon_0$$

B) Use 9.16 $\mu(T)=\varepsilon_F-\frac{\pi^2}{6}\frac{D'(\varepsilon_F)}{D(\varepsilon_F)}(k_BT)^2=\varepsilon_F-\frac{\pi^2}{6}\frac{(1/\varepsilon_0^2)}{(\varepsilon_F/\varepsilon_0^2)}(k_BT)^2=\varepsilon_F\left[1-\frac{\pi^2}{6}\left(\frac{k_BT}{\varepsilon_F}\right)^2\right]$.

Use 9.16, $\mu(T)=\varepsilon_F-\frac{\pi^2}{6}\frac{D'(\varepsilon_F)}{D(\varepsilon_F)}(k_BT)^2=\varepsilon_F-\frac{\pi^2}{6}\frac{(1/\varepsilon_0^2)}{(\varepsilon_F/\varepsilon_0^2)}(k_BT)^2=\varepsilon_F\left[1-\frac{\pi^2}{6}\left(\frac{k_BT}{\varepsilon_F}\right)^2\right]$.

Alternative Solution:

$$N=\int_0^{\infty}\frac{D(\varepsilon)d\varepsilon}{z^{-1}\exp(\beta\varepsilon)+1}=\frac{(k_BT)^2}{\varepsilon_0^2}\frac{1}{\Gamma(2)}\int_0^{\infty}\frac{x^{2-1}dx}{z^{-1}\exp(x)+1}=\left(\frac{k_BT}{\varepsilon_0}\right)^2 f_2(z).$$

Use $f_v(z)=\frac{\xi^v}{\Gamma(v+1)}\left[1+v(v-1)\frac{\pi^2}{6}\frac{1}{\xi^2}+\ldots\right]\rightarrow f_2(z)=\frac{(\mu/k_BT)^2}{2!}\left[1+\frac{\pi^2}{3}\left(\frac{k_BT}{\mu}\right)^2+..\right]$.

Solving, $\left(\frac{\mu}{k_BT}\right)^2=2f_2(z)\left[1+\frac{\pi^2}{3}\left(\frac{k_BT}{\mu}\right)^2+..\right]^{-1}=2N\left(\frac{\varepsilon_0}{k_BT}\right)^2\left[1+\frac{\pi^2}{3}\left(\frac{k_BT}{\mu}\right)^2+..\right]^{-1}$,

$$\mu=\sqrt{2N}\varepsilon_0\left[1+\frac{\pi^2}{3}\left(\frac{k_BT}{\mu}\right)^2+..\right]^{-1/2}=\varepsilon_F\left[1+\frac{\pi^2}{3}\left(\frac{k_BT}{\mu}\right)^2+..\right]^{-1/2}.$$

Now Taylor expand to get $\mu=\varepsilon_F\left[1-\frac{\pi^2}{6}\left(\frac{k_BT}{\varepsilon_F}\right)^2\right]$.

C) Use (9.12), $E_{gs}=\int_0^{\varepsilon_F}\varepsilon D(\varepsilon)d\varepsilon=\int_0^{\varepsilon_F}\frac{\varepsilon^2}{\varepsilon_0^2}d\varepsilon=\frac{\varepsilon_F^3}{3\varepsilon_0^2}$, and part A, $\frac{\varepsilon_F^2}{\varepsilon_0^2}=2N\rightarrow E_{gs}=\frac{2}{3}N\varepsilon_F$.

From 9.18, $\langle E\rangle=E_{gs}+\frac{\pi^2}{6}D(\varepsilon_F)(k_BT)^2=\frac{2}{3}N\varepsilon_F+\frac{\pi^2}{6}\frac{\varepsilon_F^2}{\varepsilon_0^2}\frac{\varepsilon_F^2}{\varepsilon_F^3}(k_BT)^2=\frac{2}{3}N\varepsilon_F\left(1+\frac{\pi^2}{2}\left(\frac{k_BT}{\varepsilon_F}\right)^2\right)$.

From eqn 9.19, $\frac{C_V}{k_B}=\frac{\pi^2}{3}D(\varepsilon_F)k_B T=\frac{\pi^2}{3}\frac{\varepsilon_F}{\varepsilon_0^2}k_B^2 T=\frac{\pi^2}{3}\left(\frac{\varepsilon_F}{\varepsilon_0}\right)^2\left(\frac{T}{\varepsilon_F/k_B}\right)=\frac{\pi^2}{3}\left(\frac{\varepsilon_F}{\varepsilon_0}\right)^2\left(\frac{T}{T_F}\right)$.

Using $\varepsilon_F=\sqrt{2N}\varepsilon_0$, $\frac{C_V}{k_B}=\frac{2}{3}N\pi^2\left(\frac{T}{T_F}\right)$.

Note the question ask that the answers must be expressed in terms k_B, N, ε_0, and T. The above forms are ok if you show how the expressions are transformed into $\varepsilon_F=\sqrt{2N}\varepsilon_0$.

Alternative Solution:

$$\ln\Xi=\frac{PV}{k_B T}=\int_0^\infty d\varepsilon D(\varepsilon)\ln\left(1+z\exp(-\beta\varepsilon)\right)=\frac{1}{\varepsilon_0^2}\int_0^\infty d\varepsilon\varepsilon\ln\left(1+z\exp(-\beta\varepsilon)\right)$$

Integrating by parts $\ln\Xi=\frac{1}{2\varepsilon_0^2}\beta\int_0^\infty\frac{\varepsilon^2 d\varepsilon}{z^{-1}\exp(\beta\varepsilon)+1}=\frac{1}{2}\left(\frac{k_B T}{\varepsilon_0}\right)^2\frac{2}{\Gamma(3)}\int_0^\infty\frac{x^{3-1}dx}{z^{-1}\exp(x)+1}$,

$\ln\Xi=\left(\frac{k_B T}{\varepsilon_0}\right)^2 f_3(z)=(\beta\varepsilon_0)^{-2}f_3(z)$. Use $\langle E\rangle=-\left(\frac{\partial\ln\Xi}{\partial\beta}\right)=2k_B T\left(\frac{k_B T}{\varepsilon_0}\right)^2 f_3(z)$

From earlier $N=\left(\frac{k_B T}{\varepsilon_0}\right)^2 f_2(z)\rightarrow\langle E\rangle=2Nk_B T\frac{f_3(z)}{f_2(z)}$. and

$$f_2(z)=\frac{(\mu/k_B T)^2}{2!}\left[1+\frac{\pi^2}{3}\left(\frac{k_B T}{\mu}\right)^2+..\right],\ f_3(z)=\frac{(\mu/k_B T)^3}{3!}\left[1+\pi^2\left(\frac{k_B T}{\mu}\right)^2+..\right]$$

$$\langle E\rangle=2Nk_B T\frac{f_3(z)}{f_2(z)}=2Nk_B T\frac{(\mu/k_B T)}{3}\left[1+\pi^2\left(\frac{k_B T}{\mu}\right)^2+..\right]\left[1+\frac{\pi^2}{3}\left(\frac{k_B T}{\mu}\right)^2+..\right]^{-1}.$$

Now use the result of B, $\mu=\varepsilon_F\left[1-\frac{\pi^2}{6}\left(\frac{k_B T}{\varepsilon_F}\right)^2\right]$.

$\langle E\rangle=\frac{2}{3}Nk_B T\frac{\varepsilon_F}{k_B T}\left[1-\frac{\pi^2}{6}\left(\frac{k_B T}{\varepsilon_F}\right)^2\right]\left[1+\pi^2\left(\frac{k_B T}{\varepsilon_F}\right)^2\right]\left[1+\frac{\pi^2}{3}\left(\frac{k_B T}{\varepsilon_F}\right)^2\right]^{-1}$, where we are only including terms that will contribute up to $(k_B T)^2$. Now Taylor expand

$$\langle E\rangle=\frac{2}{3}N\varepsilon_F\left[1+\left(-\frac{\pi^2}{6}+\pi^2-\frac{\pi^2}{3}\right)\left(\frac{k_B T}{\varepsilon_F}\right)^2\right]=\frac{2}{3}N\varepsilon_F\left[1+\frac{\pi^2}{2}\left(\frac{k_B T}{\varepsilon_F}\right)^2\right]$$

$$C_V=\left(\frac{\partial\langle E\rangle}{\partial T}\right)_{N,V}=\frac{2}{3}\pi^2 N\frac{k_B^2 T}{\varepsilon_F}\rightarrow\frac{C_V}{k_B}=\frac{2}{3}N\pi^2\left(\frac{T}{T_F}\right).$$

Problem 2) Problem 6 Chapter 9 **(10 points)**.

For a system where fermions have energy $\varepsilon_\alpha = \alpha\varepsilon_0$, with α a **positive integer**.

A) It is clear that $\varepsilon_F = N\varepsilon_0$. For N = 10^{20}, $\varepsilon_0 = 10^{-30}J$, $\varepsilon_F = 10^{-18}J = 6.25eV$.

B) At T = 300, $k_BT = 8.617\times10^{-5}eV/K\times300K = 0.026eV \ll \varepsilon_F$. Basically, $T \ll T_F$ so this is a quantum gas. It is clear that just like the 1D harmonic the density of state is constant and is simply $D(\varepsilon) = 1/\varepsilon_0$.

Use equation 9.12, $E_{gs} = \int_0^{\varepsilon_F} \varepsilon D(\varepsilon)d\varepsilon = \int_0^{\varepsilon_F} \frac{\varepsilon}{\varepsilon_0}d\varepsilon = \frac{\varepsilon_F^2}{2\varepsilon_0} = \frac{N}{2}\varepsilon_F$. From 9.18,

$$\langle E\rangle = E_{gs} + \frac{\pi^2}{6}D(\varepsilon_F)(k_BT)^2 = \frac{1}{2}N\varepsilon_F + \frac{\pi^2}{6}\frac{N}{N}\frac{1}{\varepsilon_0}\frac{\varepsilon_F}{\varepsilon_F}(k_BT)^2 = \frac{N\varepsilon_F}{2}\left(1+\frac{\pi^2}{3}\left(\frac{k_BT}{\varepsilon_F}\right)^2\right).$$

From equation 9.19, $\frac{C_V}{k_B} = \frac{\pi^2}{3}D(\varepsilon_F)k_B^2T = \frac{\pi^2}{3}N\frac{k_BT}{\varepsilon_F} = \frac{N\pi^2}{3}\left(\frac{T}{T_F}\right)$.

Alternative Solution for B

$$N = \int_0^\infty \frac{D(\varepsilon)d\varepsilon}{z^{-1}\exp(\beta\varepsilon)+1} = \frac{(k_BT)}{\varepsilon_0}\frac{1}{\Gamma(1)}\int_0^\infty \frac{x^{1-1}dx}{z^{-1}\exp(x)+1} = \left(\frac{k_BT}{\varepsilon_0}\right)f_1(z).$$

Use $f_v(z) = \frac{\xi^v}{\Gamma(v+1)}\left[1+v(v-1)\frac{\pi^2}{6}\frac{1}{\xi^2}+\ldots\right] \to f_1(z) = \frac{(\mu/k_BT)}{1!}[1+..]$. From a) $\varepsilon_F = N\varepsilon_0$

$$N = \frac{k_BT}{\varepsilon_0}f_1(z) = \frac{k_BT}{\varepsilon_0}\left(\frac{\mu}{k_BT}\right) \to \mu = N\varepsilon_0 = \varepsilon_F.$$

$$\ln\Xi = \frac{PV}{k_BT} = \int_0^\infty d\varepsilon D(\varepsilon)\ln(1+z\exp(-\beta\varepsilon)) = \frac{1}{\varepsilon_0}\int_0^\infty d\varepsilon\varepsilon\ln(1+z\exp(-\beta\varepsilon))$$

Integrating by parts $\ln\Xi = \frac{1}{\varepsilon_0}\beta\int_0^\infty \frac{\varepsilon^1 d\varepsilon}{z^{-1}\exp(\beta\varepsilon)+1} = \left(\frac{k_BT}{\varepsilon_0}\right)\frac{1}{\Gamma(2)}\int_0^\infty \frac{x^{2-1}dx}{z^{-1}\exp(x)+1}$,

$\ln\Xi = \left(\frac{k_BT}{\varepsilon_0}\right)f_2(z) = (\beta\varepsilon_0)^{-1}f_2(z)$. Use $\langle E\rangle = -\left(\frac{\partial\ln\Xi}{\partial\beta}\right) = k_BT\left(\frac{k_BT}{\varepsilon_0}\right)f_2(z)$

From earlier $N = \left(\frac{k_BT}{\varepsilon_0}\right)f_1(z) \to \langle E\rangle = Nk_BT\frac{f_2(z)}{f_1(z)}$. Use $f_1(z) = \left(\frac{\mu}{k_BT}\right)$

$f_2(z) = \frac{(\mu/k_BT)^2}{2!}\left[1+\frac{\pi^2}{3}\left(\frac{k_BT}{\mu}\right)^2+..\right]$. $\langle E\rangle = Nk_BT\frac{\mu}{2k_BT}\left[1+\frac{\pi^2}{3}\left(\frac{k_BT}{\mu}\right)^2+..\right]$, and using

$$\mu = \varepsilon_F \to \langle E\rangle = \frac{N\varepsilon_F}{2}\left[1+\frac{\pi^2}{3}\left(\frac{k_BT}{\varepsilon_F}\right)^2\right], C_V = \left(\frac{\partial\langle E\rangle}{\partial T}\right)_{N,V} \to \frac{C_V}{k_B} = \frac{N\pi^2}{3\varepsilon_F}\left(\frac{T}{\varepsilon_F/k_B}\right) = \frac{N\pi^2}{3}\left(\frac{T}{T_F}\right).$$

Problem 3) Problem 7 Chapter 9 **(10 points)**
Heating by **adiabatic expansion**. Initially, an ideal gas of N fermions is confined, by an **insulated wall** that does not allow heat exchanged, to a volume, V_i, and has, effectively, temperature zero. The volume is increased adiabatically (no heat added or removed) suddenly to V_F. When the gas comes to equilibrium it is a classical ideal gas.
A) What is the **final temperature** of the gas?
From 9.12, energy is $\langle E\rangle = \frac{3}{5}N\varepsilon_F$, and 9.10,

$$\varepsilon_F = \frac{h^2}{2m}\left(\frac{3}{8\pi}\right)^{2/3}\left(\frac{N}{V_i}\right)^{2/3} \rightarrow \langle E\rangle = \frac{3}{5}\frac{N^{5/3}}{V_i^{2/3}}\frac{h^2}{2m}\left(\frac{3}{8\pi}\right)^{2/3}.$$

At final equilibrium, since no work is done and heat is not exchanged, the equipartition theorem gives, $\langle E\rangle = \frac{3}{2}Nk_BT_F = \frac{3}{5}\frac{N^{5/3}}{V_i^{2/3}}\frac{h^2}{2m}\left(\frac{3}{8\pi}\right)^{2/3} \rightarrow T_F = \frac{2}{5}\frac{h^2}{2m}\left(\frac{3}{8\pi}\right)^{2/3}\left(\frac{N}{V_i}\right)^{2/3}$, which is independent of the final volume, V_F.
B) Derive an equation for the ratio, V_F/V_i and explain your reasoning?
In class we discussed the Sackur-Tetrode equation that shows that the change in entropy of a classical ideal gas is $\Delta S = S_F - S_i = Nk_B\ln\left(\left(\frac{V_F}{V_i}\right)\left(\frac{T_F}{T_i}\right)^{3/2}\right)$, which for a classical **adiabatic** process, the **temperature** does not change, $T_F = T_i$, $\Delta S = S_F - S_i = Nk_B\ln\left(\frac{V_F}{V_i}\right)$.

Note that the textbook makes similar arguments in section 2.4 to derive equation 2.3. But for this system the **initial entropy** should be **zero**, if it is a **quantum system** at **zero temperature**, as required by the **third law** of **thermodynamics**. But if the system is a classical ideal gas, the initial entropy is **greater** than **zero**, unless the temperature is really low. This means that the actual change in entropy is $\Delta S > Nk_B\ln\left(\frac{V_F}{V_i}\right) \rightarrow$ leading to inequality $\left(\frac{V_F}{V_i}\right) < \exp\left(\frac{\Delta S}{Nk_B}\right)$, in this **adiabatic quantum** to **classical transition**, where ΔS is the actual entropy change.
C) The title of this **Heating** by **adiabatic expansion** is an oxymoron. Explain the sense in which the title is a contradiction in terms, and the sense in which the title is a legitimate use of words.
Answer: In adiabatic process, no heat are exchanged, hence it should be possible to heat the system. However it is true that there is no heat flow from the outside, or flow out of the system, so the process is a **true adiabatic process**. There also is an increase in temperature due to the transition from quantum to classical system.

Problem 4) 3D Fermi Electron Gas in the High and Low Temperature Limit: In class we showed that $\frac{N}{V}=\frac{2}{\lambda^3}f_{3/2}(z)$, $\frac{P}{k_BT}=\frac{2}{\lambda^3}f_{5/2}(z)$, $\langle E\rangle=\frac{3}{2}Nk_BT\frac{f_{5/2}(z)}{f_{3/2}(z)}$, with $\lambda=\left(h^2/(2\pi mk_BT)\right)^{1/2}$, and $z=\exp(\beta\mu)$.

A) (5 points) Eq. 7.14 gives the **classical chemical potential** is $\mu=-k_BT\ln\left(\frac{V}{N\lambda^3}\right)$, show that in the classical ideal gas limit $\mu<0$, and that for very dilute gas the chemical potential is a large negative number. Hence explain in no more than two sentences why $0<z\ll 1$ in the **classical ideal gas limit**.

In dilute limit, V/N is large, $\frac{V}{N\lambda^3}$ is large, and $\ln\left(\frac{V}{N\lambda^3}\right)>1\rightarrow\mu=-k_BT\ln\left(\frac{V}{N\lambda^3}\right)\ll 0$.

Since $z=\exp(\beta\mu)=\exp\left(-\ln\left(\frac{V}{N\lambda^3}\right)\right)=\left(\frac{V}{N\lambda^3}\right)^{-1}\ll 1$.

B) (5 points) Show that in the **classical ideal gas limit**, the **3D Fermi Ideal gas relations** given above becomes PV = Nk$_B$T, $\langle E\rangle=(3/2)Nk_BT$, and $\mu=-k_BT\ln\left(\frac{V}{N}\frac{1}{\lambda^3}\right)$.

HINT: use properties of $f_v(z)$ for low z.

$\langle E\rangle=\frac{3}{2}Nk_BT\frac{f_{5/2}(z)}{f_{3/2}(z)}$. Using the appendix, $f_{5/2}(z)=z-\frac{z^2}{2^{5/2}}+\frac{z^3}{3^{5/2}}+\ldots\approx z$,

$f_{3/2}(z)=z-\frac{z^2}{2^{3/2}}+\frac{z^3}{3^{3/2}}+\ldots\approx z$, where $\langle E\rangle=\frac{3}{2}Nk_BT\frac{f_{5/2}(z)}{f_{3/2}(z)}\approx\frac{3}{2}Nk_BT\frac{z}{z}=\frac{3}{2}Nk_BT$.

C) (5 points) Use the free-energy relations in the appendix to show $F=\langle E\rangle-TS=N\mu-PV$.

From appendix, Helmholtz free energy $F=\langle E\rangle-TS$; Gibbs $G=F+PV=N\mu$,

$F+PV=N\mu\rightarrow\langle E\rangle-TS+PV=N\mu\rightarrow\langle E\rangle-TS=N\mu-PV$.

D) (5 points) In class we show that at **low temperature**, $\mu=\varepsilon_F\left[1-\frac{\pi^2}{12}\left(\frac{k_BT}{\varepsilon_F}\right)^2\right]$, and

$\langle E\rangle=\frac{3}{5}N\varepsilon_F\left[1+\frac{5\pi^2}{12}\left(\frac{k_BT}{\varepsilon_F}\right)^2\right]$. Use result of C to find **heat capacity** $C_V=\left(\frac{\partial\langle E\rangle}{\partial T}\right)_{N,V}$ and entropy S at low temperature up to linear order in temperature. Comment on whether your results obey the **third law** of **thermodynamics**.

$$C_V = \left(\frac{\partial\langle E\rangle}{\partial T}\right)_{N,V} = \left(\frac{\partial}{\partial T}\left(\frac{3}{5}N\varepsilon_F\left[1+\frac{5\pi^2}{12}\left(\frac{k_BT}{\varepsilon_F}\right)^2\right]\right)\right)_{N,V} \rightarrow \frac{C_V}{Nk_B} = \frac{\pi^2 k_B T}{2\varepsilon_F}$$

. Note as T→ 0, $C_V = 0$, as required by the third law of Thermodynamics.

For the **entropy**, start with, $\frac{N}{V} = \frac{2}{\lambda^3} f_{3/2}(z) \rightarrow \frac{2V}{\lambda^3} = \frac{N}{f_{3/2}}$ and

$\frac{P}{k_BT} = \frac{2}{\lambda^3} f_{5/2}(z) \rightarrow \frac{PV}{k_BT} = \left(\frac{2V}{\lambda^3}\right) f_{5/2}(z) = N\frac{f_{5/2}}{f_{3/2}}$, and $\langle E\rangle = \frac{3}{2}Nk_BT\frac{f_{5/2}(z)}{f_{3/2}(z)}$, which means

$PV = \frac{2}{3}\langle E\rangle$. From C) $\langle E\rangle - TS = N\mu - PV \rightarrow TS = \langle E\rangle + PV - N\mu \rightarrow S = \frac{5}{3}\frac{\langle E\rangle}{T} - \frac{N\mu}{T}$. Using

$\mu = \varepsilon_F\left[1-\frac{\pi^2}{12}\left(\frac{k_BT}{\varepsilon_F}\right)^2\right]$ and $\langle E\rangle = \frac{3}{5}N\varepsilon_F\left[1+\frac{5\pi^2}{12}\left(\frac{k_BT}{\varepsilon_F}\right)^2\right]$,

$S = \frac{5}{3}\frac{\langle E\rangle}{T}\frac{3}{5}N\varepsilon_F\left[1+\frac{5\pi^2}{12}\left(\frac{k_BT}{\varepsilon_F}\right)^2\right] - \frac{N}{T}\varepsilon_F\left[1-\frac{\pi^2}{12}\left(\frac{k_BT}{\varepsilon_F}\right)^2\right]$,

$\frac{S}{Nk_B} = \frac{\pi^2}{2}\frac{k_BT}{\varepsilon_F}$. Note as T→ 0, S = 0, as required by the third law of Thermodynamics.

NOTE TO MARKER: In original version, I forgot the factor N in the relation for the energy $\langle E\rangle = \frac{3}{5}N\varepsilon_F\left[1+\frac{5\pi^2}{12}\left(\frac{k_BT}{\varepsilon_F}\right)^2\right]$. Please do not take off marks from students.

Problem 5) 3D Ultra-relativistic Fermion gas with dispersion relation $\varepsilon = ap$, a is a constant. In this problem **neglect** the **spin** in **all calculations**.

A) (5 points) Show that the **density of state** is $D(\varepsilon) = V\frac{4\pi}{h^3a^3}\varepsilon^2$.

As always start with quantum counting $\sum(..) \rightarrow \frac{V}{h^3}4\pi p^2 dp$, with dispersion relation

$\varepsilon = ap \rightarrow dp = d\varepsilon/a$, $\sum(..) \rightarrow \frac{V}{h^3}4\pi p^2 dp = \frac{V}{h^3a^3}4\pi\varepsilon^2 d\varepsilon$, and $D(\varepsilon) = V\frac{4\pi}{h^3a^3}\varepsilon^2$

B) (5 points) Show $N = \int_0^\infty d\varepsilon \frac{1}{z^{-1}\exp(\beta\varepsilon)+1}D(\varepsilon) \rightarrow \frac{N}{V} = b\left(\frac{1}{\beta}\right)^{m_1} f_{m_2}(z)$, where b, m_1 and m_2 are constants that you are expected to determine.

$N = V\frac{4\pi}{h^3a^3}\int_0^\infty d\varepsilon \frac{\varepsilon^2}{z^{-1}\exp(\beta\varepsilon)+1}$. Write $x = \beta\varepsilon$,

$N = 4\pi V\left(\frac{k_BT}{ha}\right)^3 \frac{2!}{\Gamma(3)}\int_0^\infty d\varepsilon \frac{x^{3-1}}{z^{-1}\exp(x)+1} = 8\pi V\left(\frac{k_BT}{ha}\right)^3 f_3(z)$. Hence $m_1 = m_2 = 3$, and

$b = \frac{8\pi V}{h^3a^3}$.

C) (**5 points**) The heat capacity of the **3D Ultra-relativistic Fermion gas** are

$\frac{C_V}{Nk_B} = 12\frac{f_4(z)}{f_3(z)} - 9\frac{f_3(z)}{f_2(z)}$, $\frac{C_P}{Nk_B} = -12\frac{f_4(z)}{f_3(z)} + 16\frac{f_2(z)(f_4(z))^2}{(f_3(z))^3}$ (**original version was wrong**).Using the properties of the Fermi-Dirac Function given in the appendix show that the **heat capacities** obey the **third law of thermodynamics**. **HINT:** Use low-temperature expansion in the appendix, and since z is **large**, keep **only** the **first term** in the **expansion**.

From appendix, $f_\nu(z) = \frac{\xi^\nu}{\Gamma(\nu+1)}\left[1+\nu(\nu-1)\frac{\pi^2}{6}\frac{1}{\xi^2}+\ldots\right]$. At very low T, keep only the first term, $f_2 = \frac{\xi^2}{2}; f_3 = \frac{\xi^3}{3!}; f_4 = \frac{\xi^4}{4!}, \xi = \beta\mu$.

$\frac{C_V}{Nk_B} = 12\frac{f_4(z)}{f_3(z)} - 9\frac{f_3(z)}{f_2(z)} = 12\frac{(\xi^4/24)}{(\xi^3/6)} - 9\frac{(\xi^3/6)}{(\xi^2/2)} = 0$, as required by third law of thermodynamics.

NOTE TO MARKER: Due to error do not grade the part below on C_P.

For $\frac{C_P}{Nk_B} = -12\frac{f_4(z)}{f_3(z)} + 16\frac{f_2(z)(f_4(z))^2}{(f_3(z))^3} = -12\frac{(\xi^4/24)}{(\xi^3/6)} + 16\frac{(\xi^2/2)(\xi^4/24)^2}{(\xi^3/6)^3} = 0$

Appendix from "Statistical Mechanics", 3rd edition, Pathria and Beale

Grand-canonical Ensemble

Grand partition function, $\Xi = \sum_i z^{N_i}\exp(-\beta E_i)$, $z = \exp(\beta\mu)$.

Probability of state i with energy E_i and particle number N_i, $P_i = \frac{z^{N_i}\exp(-\beta E_i)}{\Xi}$.

Average Energy $\langle E\rangle = -\left(\frac{\partial \ln\Xi}{\partial\beta}\right)$ and **average particle number** $\langle N_i\rangle = N = z\frac{\partial\ln\Xi}{\partial z}$.

Grand Potential $\Omega_g = -k_BT\ln\Xi = -PV$

Free Energy Helmoltz $F = \langle E\rangle - TS$; Gibbs $G = F + PV = N\mu$; $\Omega_g = F - N\mu$.

Shannon's Theorem $S=-k_B\sum_i p_i \ln p_i$.

Fermi-Dirac (FD) Degenerate gas

Average number of particle in state of energy ε, $\langle n\rangle_{FD}=\dfrac{1}{z^{-1}\exp(\beta\varepsilon)+1}$, $z=\exp(\beta\mu)$.

$$N=\sum_\varepsilon \frac{1}{z^{-1}\exp(\beta\varepsilon)+1}=\int_0^\infty d\varepsilon \frac{1}{z^{-1}\exp(\beta\varepsilon)+1}D(\varepsilon)\,,\ \frac{PV}{k_BT}=\ln\Xi=\sum_\varepsilon \ln\left(1+z\exp(-\beta\varepsilon)\right).$$

$$\frac{PV}{k_BT}=\int_0^\infty d\varepsilon D(\varepsilon)\ln\left(1+z\exp(-\beta\varepsilon)\right).$$

Fermi-Dirac Functions $f_v(z)=\dfrac{1}{\Gamma(v)}\displaystyle\int_0^\infty \frac{x^{v-1}dx}{z^{-1}\exp(x)+1}$

$$\Gamma(v)=(v-1)!=\int_0^\infty \exp(-x)x^{v-1}\,dx\ ,\Gamma(m+1)=m!,\ m=0,1,2....,0!=1\,;$$

$$\Gamma\left(m+\frac{1}{2}\right)=\frac{1\bullet3\bullet5....\bullet(2m-1)}{2^m}\sqrt{\pi}\,,m=1,2,3...\ ,\Gamma(1/2)=\sqrt{\pi}\ .$$

Useful for **classical limit**, **low z**, $f_v(z)=z-\dfrac{z^2}{2^v}+\dfrac{z^3}{3^v}-\dfrac{z^4}{4^v}+....$.

Low temperature limit, $z=\exp(\beta\mu)\,,\xi=\beta\mu=\ln z$:

$$f_v(z)=\frac{\xi^v}{\Gamma(v+1)}\left[1+v(v-1)\frac{\pi^2}{6}\frac{1}{\xi^2}+v(v-1)(v-2)(v-3)\frac{7\pi^4}{360}\frac{1}{\xi^4}+...\right].$$

Using $\xi^v=(\ln z)^v$, some values are,

v = 5/2, $f_{5/2}(z)=\dfrac{8}{15\pi^{1/2}}(\ln z)^{5/2}\left[1+\dfrac{5\pi^2}{8}\dfrac{1}{(\ln z)^2}+....\right]$,

v = 3/2, $f_{3/2}(z)=\dfrac{4}{3\pi^{1/2}}(\ln z)^{3/2}\left[1+\dfrac{\pi^2}{8}\dfrac{1}{(\ln z)^2}+....\right]$,

v = 1/2, $f_{1/2}(z)=\dfrac{2}{\pi^{1/2}}(\ln z)^{5/2}\left[1-\dfrac{\pi^2}{24}\dfrac{1}{(\ln z)^2}+....\right]$,