QUESTION: Part I, A system of dilute molecular gas has entropy $S = k\ln\left[C\left(\frac{V}{N}\right)^{N}\left(\frac{E}{N}\right)^{N\frac{f}{2}}\right]$, where f is the **quadratic degrees** of **freedom** of the molecule, and C is a constant that depends on molecular details. Using the appropriate partial derivative, find temperature as a function of N, V, E: $T(N,V,E)$.

Solution Part I: Here use the definition of equations (4.22) $\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{N,V}$. Since the partial derivative is taken while N and V are held constant, it is simplest to rewrite the entropy as $S = \frac{Nfk}{2}\ln E + g(N,V)$, where $g(N,V)$ is a function of N and V only. This gives

$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{N,V} = \frac{Nfk}{2}\frac{1}{E}$, which rearranges to $E = \frac{Nfk}{2}T$, which is basically the equipartition theory.

QUESTION Part II, Instead of writing the **entropy** $S(N,E,V)$ as a function of N, E and V, write the **internal energy** $E(N,S,V)$, as a function of N, S and V. Use **differential calculus** and the **first law** of Thermodynamics to show $\left(\frac{\partial E}{\partial S}\right)_{N,V} = T$.

Solution Part II: $E(N,S,V) \to dE = \left(\frac{\partial E}{\partial S}\right)_{N,V} dS + \left(\frac{\partial E}{\partial V}\right)_{N,S} dV + \left(\frac{\partial E}{\partial N}\right)_{S,V} dN$

From first law, $dE = TdS - PdV + \mu dN$. Comparing the two equations, we note that the coefficient of dS gives $\left(\frac{\partial E}{\partial S}\right)_{N,V} = T$.

QUESTION Part III: Invert the entropy of part b) to express $E(N,S,V)$, then use the result of part c to find the equipartition theorem, $E = \frac{Nf}{2}k_B T$.

Solution Part III: $S = k_B \ln\left[C\left(\frac{V}{N}\right)^{N}\left(\frac{E}{N}\right)^{N\frac{f}{2}}\right] \to \frac{1}{C}\exp\left(\frac{S}{Nk_B}\right) = \left(\frac{V}{N}\right)\left(\frac{E}{N}\right)^{\frac{f}{2}}$

$E(S,N,V) = N\left(\frac{N}{VC}\right)^{2/f}\exp\left(\frac{2S}{fNk_B}\right)$. Now take the partial derivative

$$\left(\frac{\partial E}{\partial S}\right)_{N,V} = N\left(\frac{N}{VC}\right)^{2/f}\left[\frac{\partial}{\partial S}\exp\left(\frac{2S}{fNk_B}\right)\right]_{N,V} = \frac{2}{fNk_B}\left\{N\left(\frac{N}{VC}\right)^{2/f}\exp\left(\frac{2S}{fNk_B}\right)\right\}.$$

Buy the term in the curly bracket is simply E, so $\left(\frac{\partial E}{\partial S}\right)_{N,V} = \frac{2}{fNk_B} E = T$, which leads to the equipartition theorem, $E = \frac{Nf}{2} k_B T$.