Identical Particles, Symmetry, and Pauli Exclusion Principle
PHYS2332 Modern Physics II W2018

Identical Particles Consider the 1D harmonic oscillator as a quasi-particle (i.e. an excitation that can be treated like a particle), with solution given by the *wave* function ψ_n (equation 6.57 on page 223) that describes the n(= 0,1,2…) state with energy $E_n = \hbar\omega(n+1/2)$, equation 6.58 (page 223). Assume that there are two such **identical particles or quasi-particles**: particle number 1 in the nth quantum state $\psi_n(1)$; particle number 2 in the mth quantum state $\psi_m(2)$.

Definition: let $\psi_m(x_1)$ be the **normalized** wavefunction in equation 6.57 that described particle number 1 in a 1D harmonic oscillator state m, with energy $E_m = \hbar\omega(m+1/2), m = 0,1,2,3...$. Similarly, let $\psi_n(x_2)$ be the **normalized** wavefunction of particle number 2 in a 1D harmonic oscillator state n, with energy $E_n = \hbar\omega(n+1/2), n = 0,1,2,3...$.

Finally let $\psi(1,2)$ denotes the total wavefunction that describes the system of **two identical particles**.

a) Assume the particles are **distinguishable**. Determine the energy of the **ground state** and **first-excited state**, and write down the **normalized** form of the corresponding two-particle wave functions $\psi(1,2)$.

Ground (0): $\psi_0(1,2) = \psi_0(x_1)\psi_0(x_2)$, particle 1 in m = 0 state with energy $\hbar\omega/2$, and particle 2 in n = 0 state with energy $\hbar\omega/2$. The ground state energy is hence $\hbar\omega/2 + \hbar\omega/2 = \hbar\omega$.

First Excited (1): $\psi_1(1,2) = \psi_0(x_1)\psi_1(x_2)$, particle 1 in m = 0 state with energy $\hbar\omega/2$, and particle 2 in n = 1 state with energy $3\hbar\omega/2$. The first excited state energy is hence $\hbar\omega/2 + 3\hbar\omega/2 = 2\hbar\omega$. **Alternatively**, we can switched the roles of particle 1 and 2, by exchanging the particle label $1 \leftrightarrow 2$, so that particle 1 is in state m = 1, and particle 2 is in the state n = 0.

b) Assume the particles are **indistinguishable bosons**. Determine the energy of the **ground state** and **first-excited state**, and write down the **normalized** form of the corresponding two-particle wave functions $\psi(1,2)$.

Ground (0): $\psi_0(1,2) = \psi_0(x_1)\psi_0(x_2)$, particle 1 in m = 0 state with energy $\hbar\omega/2$, and particle 2 in n = 0 state with energy $\hbar\omega/2$. Hence the ground state energy is $\hbar\omega/2 + \hbar\omega/2 = \hbar\omega$. **Note** that there's no difference with the distinguishable particle of part b, since identical bosons are allowed to occupy the same quantum states.

First Excited (1): $\psi_1(1,2) = \frac{1}{\sqrt{2}}\left(\psi_0(x_1)\psi_1(x_2) + \psi_1(x_1)\psi_0(x_2)\right)$. The first term indicates particle 1 in m = 0 state with energy $\hbar\omega/2$, and particle 2 in n = 1 state with energy $3\hbar\omega/2$. The second term indicates particle 1 in m = 1 state with energy $3\hbar\omega/2$, and particle 2 in n = 0 state with energy $\hbar\omega/2$. In both cases, the first excited state energy is hence $\hbar\omega/2 + 3\hbar\omega/2 = 2\hbar\omega$. **Note** that $\psi_1(1,2)$ expresses that one particle is in the m = 1 state and the other is in the n = 2 state, but we can't tell which ones. This is the essence of the principle of indistinguishable identical particles.

Symmetry: If the labels of particle 1 and 2 are exchanged $1 \leftrightarrow 2$, the total wavefunction becomes:

$$\psi_1(2,1) = \frac{1}{\sqrt{2}}\left(\psi_0(x_2)\psi_1(x_1) + \psi_1(x_2)\psi_0(x_1)\right)$$

$$= \frac{1}{\sqrt{2}}\left(\psi_0(x_1)\psi_1(x_2) + \psi_1(x_2)\psi_0(x_1)\right) = \psi_1(1,2)$$

Hence $\psi_1(2,1) = +\psi_1(1,2)$, and the wave function is symmetric. Similarly it is **trivial** to show that for the ground state $\psi_0(2,1) = +\psi_0(1,2)$.

c) Assume the particles are **indistinguishable fermions**. Determine the energy of the **ground state** and **first-excited state**, and write down the **normalized** form of the corresponding two-particle wave functions $\psi(1,2)$.

Unlike the previous two cases the two identical fermions obeys the **Pauli exclusion principle**, and cannot both occupy the n = 0 = m states. One of the particles must be in the 1 state, as shown below:

Ground State (0): $\psi_0(1,2) = \frac{1}{\sqrt{2}}\left(\psi_0(x_1)\psi_1(x_2) - \psi_1(x_1)\psi_0(x_2)\right)$. The first term indicates particle 1 in m = 0 state with energy $\hbar\omega/2$, and particle 2 in n = 1 state with energy $3\hbar\omega/2$. The second term indicates particle 1 in m = 1 state with energy $3\hbar\omega/2$, and particle 2 in n = 0 state with energy $\hbar\omega/2$. In both cases, the ground state energy is hence $\hbar\omega/2 + 3\hbar\omega/2 = 2\hbar\omega$.

Anti-Symmetry: If the labels of particle 1 and 2 are exchanged $1 \leftrightarrow 2$, the total wavefunction becomes:

$$\psi_0(2,1) = \frac{1}{\sqrt{2}}\left(\psi_0(x_2)\psi_1(x_1) - \psi_1(x_2)\psi_0(x_1)\right)$$

$$= -\frac{1}{\sqrt{2}}\left(\psi_0(x_1)\psi_1(x_2) - \psi_1(x_2)\psi_0(x_1)\right) = -\psi_1(1,2)$$

Hence $\psi_1(2,1) = -\psi_1(1,2)$, and the wave function is anti-symmetric.

First excited state (1): $\psi_0(1,2) = \frac{1}{\sqrt{2}}\left(\psi_0(x_1)\psi_2(x_2) - \psi_2(x_1)\psi_0(x_2)\right)$. The first term indicates particle 1 in m = 0 state with energy $\hbar\omega/2$, and particle 2 in n = 2 state with energy $5\hbar\omega/2$. The second term indicates particle 1 in m = 2 state with energy $5\hbar\omega/2$, and particle 2 in n = 0 state with energy $\hbar\omega/2$. In both cases, the first excited state energy is $\hbar\omega/2 + 5\hbar\omega/2 = 6\hbar\omega$. It is easy to show that this state is **anti-symmetric**.

Final Point: Note that the state where both particle are in the state n = 1 = m also has the energy $6\hbar\omega$, but again this is **forbidden** for **identical fermions**.