PHYS 1211 F2020 Lecture Wednesday November 4, 2020

Chapter 7

Introduction to Work by spring and variable force

Calculus for 1st year physics: Derivatives and Indefinite Integrals

- Consider a function y of a variable x, $y(x)$
- **Derivative** of y wrt x, $\frac{dy}{dx} = y^{/}$
- Example, $y = x^2 \rightarrow \frac{dy}{dx} = \frac{d(x^2)}{dx} = 2\times x^{2-1} = 2x$
- **Indefinite integral**, $\int y\, dx = z(x)$, where $y(x)$ and $z(x)$ are functions of x.
- The function $z(x)$ is the **anti-derivative** of the function $y(x)$: $\frac{dz}{dx} = y(x)$
- Example $y = x \rightarrow \int y\, dx = \int x\, dx = \frac{1}{2}x^2 = z(x)$
- Since if $z(x) = \frac{1}{2}x^2 \rightarrow \frac{dz}{dx} = \frac{d\left(\frac{1}{2}x^2\right)}{dx} = \frac{1}{2}2\times x^{2-1} = x$

Calculus for 1st year physics: Definite Integrals

- **Definite integral** $\int_{x_i}^{x_f} y(x)dx$ is **evaluated** between the **limits,** $x_i \leq x \leq x_f$
- The answer of a **definite integral** is a **numerical value**
- $\int_{x_i}^{x_f} y(x)dx = [z(x)]_{x_i}^{x_f} = z\left(x_f\right) - z(x_i)$
- Example $y = x \rightarrow \int y\, dx = \int x\, dx = \frac{1}{2}x^2 = z(x)$
- $\int_{x_i}^{x_f} x\, dx = [z(x)]_{x_i}^{x_f} = z\left(x_f\right) - z(x_i) = \frac{1}{2}x_f^2 - \frac{1}{2}x_i^2$

Chain Rule

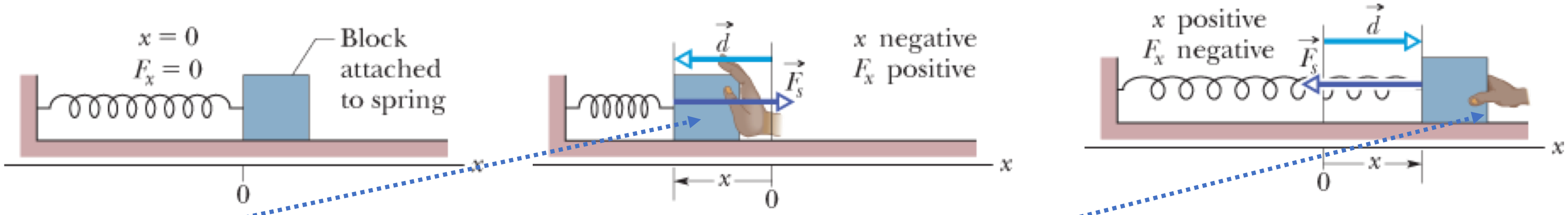
- Let x be a function of t, $x(t)$
- If y is a function of x, $y(x)$ then y is also a function of t, $y(t)$
- Now consider $\frac{dy}{dt}$, the **chain rule** states $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$
- **Example**, $x(t) = t^2, y(x) = x^2 = (t^2)^2 = t^4 = y(t)$
- Since $y(t) = t^4 \rightarrow \frac{dy}{dt} = 4\times t^{4-1} = 4t^3$
- Also x$(t) = t^2 \rightarrow \frac{dx}{dt} = 2\times t^{2-1} = 2t$
- Noting $y = x^2$, use the **chain rule** $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{d(x^2)}{dx}\times 2t = 2\times x^{2-1}\times 2t = 4xt$
- But $x(t) = t^2$
- This means $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = 4xt = 4t^3$

Advance Differential Calculus

- Let x be a function of t, $x(t)$
- If y is a function of x, $y(x)$ then y is also a function of t, $y(t)$
- The **chain rule** states $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$
- Now consider the **definite integral** $\int_{x_i}^{x_f} \frac{dy}{dt} dx$
- We can change the variable of integration form x to … say y by doing the following
- $\int_{x_i}^{x_f} y(x)\, dx = \int_{y_i}^{y_f} \frac{dy}{dx}\frac{dx}{dt} dx$, where the **integration limits** are now $y(x_i) = y_i$ and $y(x_f) = y_f$
- Though I have not proven this, $\frac{dy}{dx} dx = dy$
- $\int_{x_i}^{x_f} \frac{dy}{dt}\, dx = \int_{y_i}^{y_f} \frac{dy}{dx}\frac{dx}{dt} dx = \int_{y_i}^{y_f} \frac{dx}{dt} dy$,where $\frac{dx}{dt}$ must be expressed as a function of y

The Spring Force

Figure below shows a spring in an equilibrium (relaxed) state:



If a **leftward** force is applied to the block that **compresses** the spring, x < 0 (**left**), the spring applies a **rightward** restoring spring force, $\vec{F_S}$: $F_x > 0$ pushes on the block.

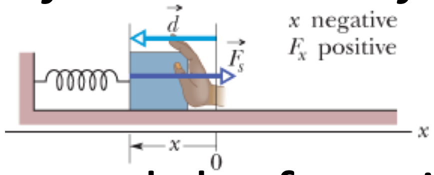
If a **rightward** force is applied to the block that **stretches** the spring , x > 0 (**right**), the spring applies a **leftward** restoring spring force, $\vec{F_S}$: $F_x < 0$ pulls the block.

Hooke's Law: Spring force $F_x = -kx$, x is displacement of spring, k is the spring constant in unit of N/m

Work by a spring

Work by **variable spring force** in 1D that changes the **compression/stretching** of **spring** from an **initial** x_i to **final** x_f.

Divide into **infinitesimal portion**, Δx_j, where the subscript *j* denotes the *j^th^* portion of the path.



Work by spring, $\mathrm{d}W_s = F_{xj}\Delta x_j > 0\ , F_{xj} < 0, \Delta x_j < 0$

It is assumed that the infinitesimal portion, Δx_j, is straight, and the force is constant, F_{xj}.

The total work done by spring on the path is found by summing all *j*th portions: $W_s = \sum dW = \sum_j F_{xj}\Delta x_j$

This gives $W_s = \int_{x_i}^{x_f} F_x(x)dx = -\int_{x_i}^{x_f} kx\ dx =?$

What is the definite integral, $\int kx\ dx =?$ $\int kx\ dx = \frac{1}{2}kx^2$

Definite integral, total work by spring, $W_s = -\int_{x_i}^{x_f} kx\ dx = \left[-\frac{1}{2}kx^2\right]_{x_i}^{x_f}$

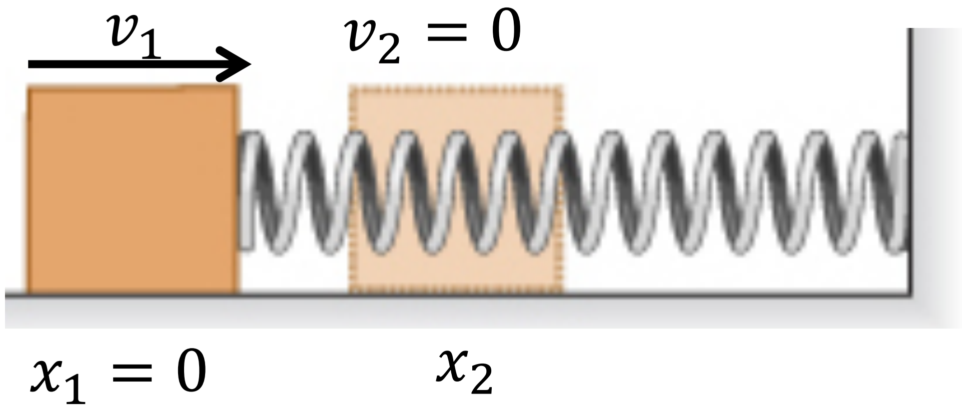
$$W_s = \left(-\frac{1}{2}kx_f^2\right) - \left(-\frac{1}{2}kx_i^2\right) = \frac{1}{2}kx_i^2 - \frac{1}{2}kx_f^2$$

Example spring

A typical spring constant is $k = 400\,\frac{N}{m}$. Below such a spring is being compressed from **equilibrium**, at $x_1 = 0$, by a m = 8 kg box, moving at $v_1 = 9\,\frac{m}{s}$. Assume no friction.

A) Find maximum compression, x_2.

B) Find the speed, v_3, when the box is at the equilibrium at $x_3 = 0$.

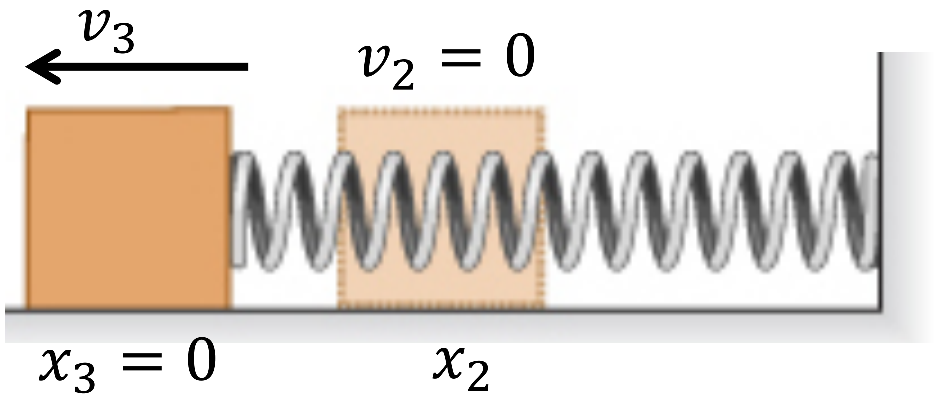


Part A

Work by spring from x_1 to x_2, $W_{12} = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2$

Work-Energy, $W_{12} = -\frac{1}{2}kx_2^2 = \Delta K = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$

$$x_2 = \sqrt{\frac{m}{k}}v_1 = \sqrt{\frac{8kg}{400Nm^{-1}}}\times 9\,\frac{m}{s} = 1.27m$$



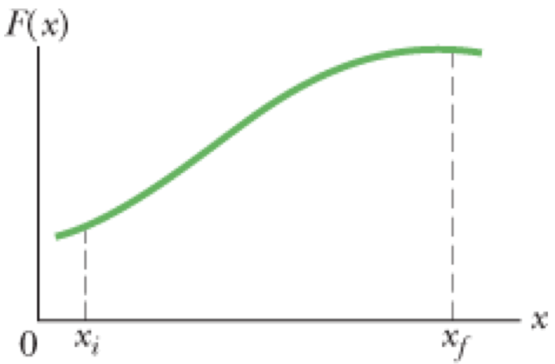
Part B: Use work-energy theorem from 2 to 3

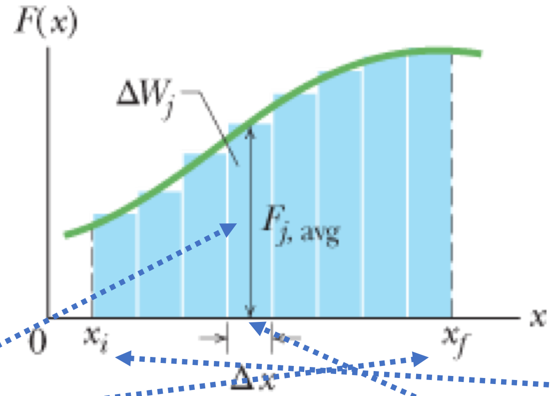
$$W_{23} = \frac{1}{2}kx_2^2 - \frac{1}{2}kx_3^2 = \Delta K = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_3^2$$

$$v_3 = \sqrt{\frac{k}{m}}x_2 = \sqrt{\frac{400Nm^{-1}}{8kg}}\times 1.27m = 8.98\,\frac{m}{s}$$

Work by general variable force in 1D

Consider a object moving in 1D acted on by a general variable force as shown below, $F(x)$.





The work done is the area below the F vs. x curve from the initial position x_i to final position x_f.

Divide into very small paths each of length, Δx, where at position x_j the average force is $F_{j,avg}$, which contributes work $\Delta W_j = F_{j,avg}\Delta x$

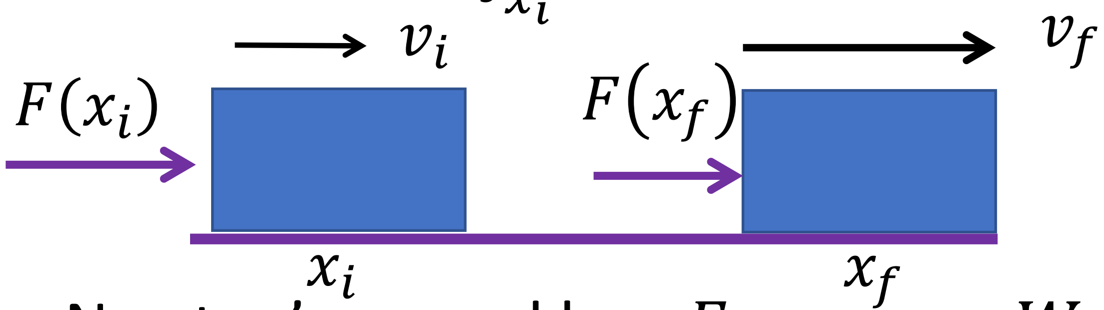
The total work done is $W \sim \sum_j W_j = \sum_j F_{j,avg}\Delta x$

In the calculus limit of very small $\Delta x \rightarrow 0$, the total work done is

$$W = \int_{x_i}^{x_f} F(x)dx$$

Work-Energy Theorem by a general variable force in 1D

The work done is $W = \int_{x_i}^{x_f} F(x)dx$.



The Use Newton's second law, $F = ma \rightarrow W = \int_{x_i}^{x_f} madx$

Divide Acceleration is $a = \frac{dv}{dt} \rightarrow adx = \frac{dv}{dt}dx$

Note velocity v is a function of position x and time t.

Use the chain rule, $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$, where $\frac{dx}{dt} = v$

$$adx = \frac{dv}{dt}dx = v\frac{dv}{dx}dx = vdv$$

$$W = \int_{x_i}^{x_f} madx = m\int_{v_i}^{v_f} vdv = \left[\frac{1}{2}mv^2\right]_{v_i}^{v_f} = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = K_f - K_i$$

Work done $W = \Delta K$ change in kinetic energy