Extrema of a function

Definitions $x(t) \equiv x$ is a function of t

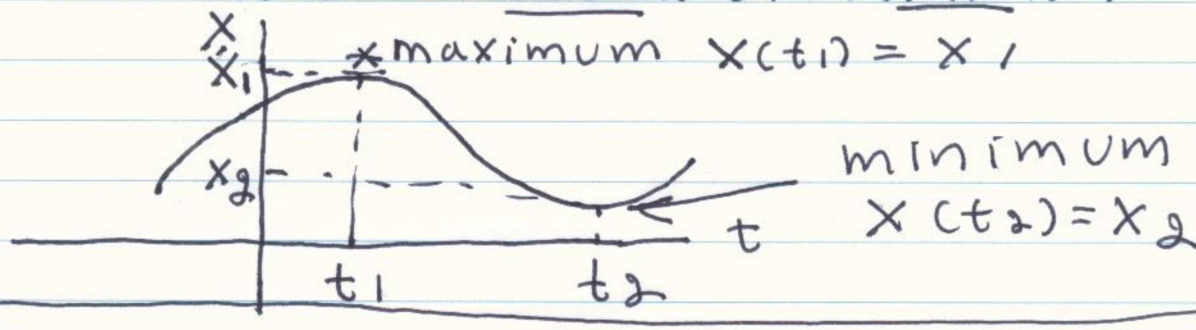
$\frac{dx}{dt} \equiv$ derivative of x with respect to (wrt) t

alternate notation $\frac{dx}{dt} = x'$

$\frac{d^2x}{dt^2} \equiv$ second (2^{nd}) derivative of x w.r.t t

also $\frac{d^2x}{dt^2} \equiv \frac{d}{dt}\left(\frac{dx}{dt}\right) \equiv x''$

extrema of $x(t)$ are the values of t when x are "local" maxima or minima:



Condition for Extremum (maximum/minimum)

$\boxed{\frac{dx}{dt} = 0} \rightarrow$ solve for t

Determining whether extremum is a Maximum or a Minimum

Let $t = t_1$ be extremum $\Rightarrow \frac{dx}{dt}(t_1) = 0$

Step 1 find $\frac{d^2x}{dt^2}(t) \equiv$ function of

Step 2 Evaluate at t_1 $\frac{d^2x}{dt^2}(t_1)$

Step 3 if $\frac{d^2x}{dt^2}(t_1) > 0$ it is a minimum

if $\frac{d^2x}{dt^2}(t_2) < 0$ " " " maximum

Example 1 Find maximum or minimum of

$$x = 3t^2 - 6t + 4$$

$$\frac{dx}{dt} = 3\frac{d(t^2)}{dt} - 6\frac{d(t)}{dt} + \cancel{\frac{d(4)}{dt}} = 6t - 6$$

$$\frac{dx}{dt} = 0 \rightarrow 6t - 6 = 0 \rightarrow 6t = 6 \rightarrow t = 1$$

extremum at $t = 1$

Is $x(1)$ a maximum or a minimum?

To answer this start with $\frac{dx}{dt} = 6t - 6$

then find $\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = 6\frac{d(t)}{dt} - \frac{d(6)}{dt}$

$$= 6 > 0$$

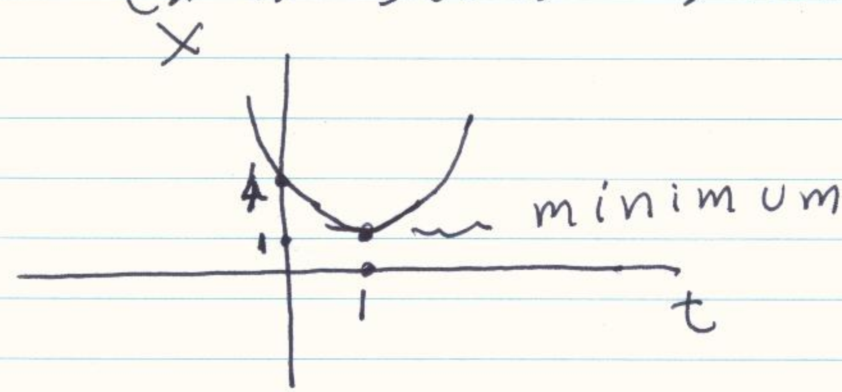
so $\frac{d^2x}{dt^2}$ positive alway, and $\frac{d^2x}{dt^2}(t=1) = 6 > 0$

So at $t=1$, x is a minimum

$$X(1) = 3(1)^2 - 6(1) + 4 = 1$$ minimum Value

Note $X = 3t^2 - 6t + 4 = 3(t-1)^2 + 1$

$(x-1) = 3(t-1)^2$ shifted parabola



Examples of indefinite Integrals

i) $f(t) = 3t^2$ find $\int f(t)\,dt$

$$\int f(t)\,dt = \int 3t^2\,dt$$

$$= 3\left(\frac{t^3}{3}\right) + C = \overbrace{t^3 + C}^{\text{RHS}}, \quad C \equiv \text{arbitrary constant}$$

IF we differentiate right Hand side (RHS)

$$\frac{d}{dt}(t^3 + C) = \frac{d(t^3)}{dt} + \frac{d(C)}{dt}$$

$$= 3t^2 + 0 \quad = f(t)$$

we obtain the <u>integrand</u> $f(t)$

ii) $f(t) = 6t + 3$

$$\int f(t)\,dt = \int 6t\,dt + \int 3\,dt$$

$$= 6\left(\frac{t^2}{2}\right) + 3t + D, \quad D \equiv \text{constant}$$

$$= \underbrace{3t^2 + 3t + D}_{\text{RHS}}$$

Differentiating RHS

$$\frac{d(3t^2 + 3t + D)}{dt} = 3\frac{d(t^2)}{dt} + 3\frac{d(t)}{dt} + \frac{dD}{dt}$$

$$= 6t + 3 + 0$$

$$= f(t)$$

DEFinite Integral Example

$f(t) = 3t^2$, find $\int_{-1}^{2} f(t)\,dt$

First Find Indefinite Integral

$$\int f(t)\,dt = g(t)$$

$$\int 3t^2\,dt = t^3 + c$$ ← from previous example

then evaluate $g(t)$ at $t=-1$ & $t=2$

$$\int_{-1}^{2} f(t)\,dt = g(t)\Big|_{t=-1}^{t=2} = g(2) - g(-1)$$

$$\int_{-1}^{2} 3t^2\,dt = (t^3 + c)\Big|_{t=-1}^{t=2}$$

$$= (2^3 + \cancel{c}) - ((-1)^3 + \cancel{c})$$

$$= 9$$

Note that constant c does not matter, so can set to zero

$$\int_{-1}^{2} 3t^2\,dt = t^3\Big|_{t=-1}^{t=2} = (2)^3 - (-1)^3$$

$$= 9$$